## Section 6.3

In trigonometric equations, sometimes a multiple or partial angle occurs such as $\cos(2\theta)$ or $\tan\left(\frac{x}{2}\right)$. Furthermore, some equations may contain more than one trigonometric function.

Not always, but in general, it is best to try to convert the equation to one involving a single trigonometric of the same angle. Even having different powers of the same function is not necessarily a problem.

1. Sometimes a multiple angle can be eliminated.
   Solve the following for x in radians ($0 \le x < 2\pi$):
   $\cos(2x) = \cos(x) \Rightarrow 2\cos^2(x) - 1 = \cos(x) \Rightarrow 2\cos^2(x) - \cos(x) - 1 = 0 \Rightarrow$
   $(2\cos(x)+1)(\cos(x)-1) = 0 \Rightarrow 2\cos(x) + 1 = 0$ or $\cos(x) - 1 = 0 \Rightarrow$
   $\cos(x) = -\frac{1}{2}$ or $\cos(x) = 1 \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$ or $x = 0$ (radians) using the table of exact values. Therefore the solutions in $0 \le x < 2\pi$ are $\frac{2\pi}{3}, \frac{4\pi}{3}, 0$.

   **Note**: Using the double angle identity for cosine, we converted $\cos(2x)$ to $2\cos^2(x) - 1$, thus eliminating the multiple angle 2x.

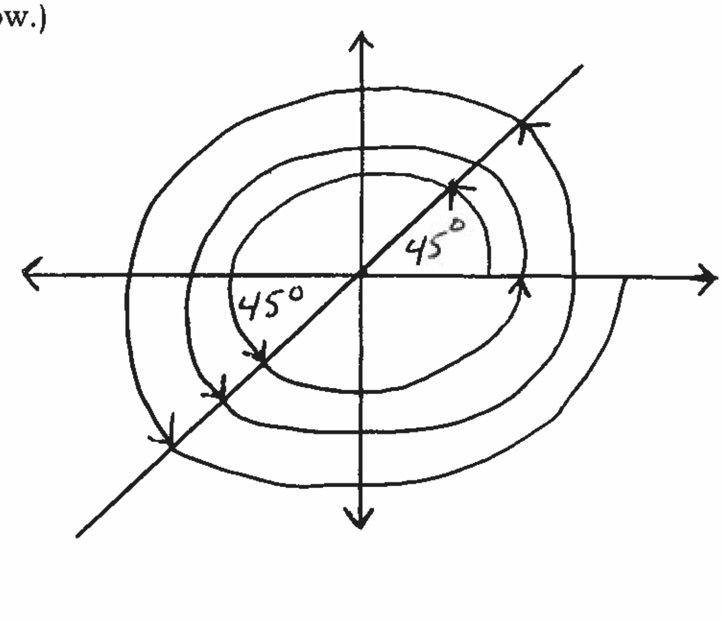
2. Sometimes it is easier to *keep* a multiple angle.

   Solve the following for θ in degrees ($0° \le \theta < 360°$):
   $\tan(3\theta) - 1 = 0 \Rightarrow \tan(3\theta) = 1$.

   But $0° \le \theta < 360° \Rightarrow 0° \le 3\theta < 1080°$, so we locate the solutions for $0° \le \theta < 360°$, and then measure these angles counter-clockwise from the positive x-axis, going around the Cartesian Plane 3 times since $0° \le 3\theta < 1080°$. (See the diagram below.)

   Thus $\tan(3\theta) = 1 \Rightarrow$
   $3\theta = 45°, 225°, 405°, 585°, 765°, 945°$
   using the table of exact values.

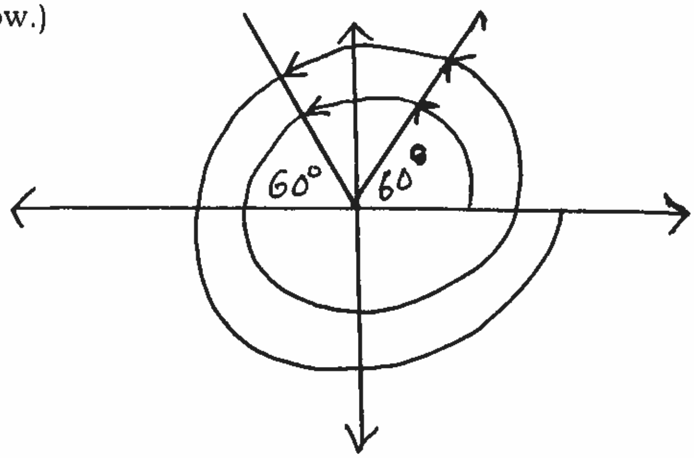
   Dividing both sides by 3 yields
   $\theta = 15°, 75°, 135°, 195°, 255°, 315°$,
   all of which satisfy $0° \le \theta < 360°$.



3. Sometimes it is actually easier to convert an equation without a multiple angle to one that contains a multiple angle using the technique in the preceding example.

Solve the following for $\theta$ in degrees ($0° \le \theta < 360°$):
$4\sin(\theta)\cos(\theta) - \sqrt{3} = 0 \Rightarrow 4\sin(\theta)\cos(\theta) = \sqrt{3} \Rightarrow 2\cdot[2\sin(\theta)\cos(\theta)] = \sqrt{3} \Rightarrow$
$2\sin(2\theta) = \sqrt{3} \Rightarrow \sin(2\theta) = \dfrac{\sqrt{3}}{2}$.

But $0° \le \theta < 360° \Rightarrow 0° \le 2\theta < 720°$, so we locate the solutions for $0° \le \theta < 360°$, and then measure these angles counter-clockwise from the positive x-axis, going around the Cartesian Plane <u>2 times</u> since $0° \le 2\theta < 720°$. (See the diagram below.)




Thus $\sin(2\theta) = \dfrac{\sqrt{3}}{2} \Rightarrow$
$2\theta = 60°, 120°, 420°, 480°$
using the table of exact values.

Dividing both sides by 2 yields
$\theta = 30°, 60°, 210°, 240°$,
<u>all of which satisfy $0° \le \theta < 360°$</u>.

4. Sometimes an equation involves a partial angle rather than a multiple angle. In this case, the same procedure is used as in the case of a multiple angle.

Solve the following for θ in degrees ($0° \le \theta < 360°$):

$$2\cos\left(\frac{\theta}{2}\right) - 1 = 0 \Rightarrow 2\cos\left(\frac{\theta}{2}\right) = 1 \Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{1}{2}.$$

But $0° \le \theta < 360° \Rightarrow 0° \le \frac{\theta}{2} < 180°$, so we locate the solutions for $0° \le \theta < 360°$, and then measure these angles counter-clockwise from the positive x-axis, going around the Cartesian Plane <u>halfway</u> since $0° \le \frac{\theta}{2} < 180°$. (See the diagram below.)

Thus $\cos\left(\frac{\theta}{2}\right) = \frac{1}{2} \Rightarrow \frac{\theta}{2} = 60°, 300°$
using the table of exact values.

Multiplying both sides by 2 yields $\theta = 120°$, <u>which satisfies $0° \le \theta < 360°$</u>.

**Note**: Although 2 solutions were initially identified in the Cartesian Plane, we used the only one that was located within half way around the plane counter-clockwise from the positive x-axis. Had we used the angle 300° in the fourth quadrant, then multiplying both sides of the equation by 2 would yield $\theta = 600°$, <u>which does NOT satisfy $0° \le \theta < 360°$</u>.

In the following example we illustrate two changes from the previous examples. Instead of solving in degrees ($0° \le \theta < 360°$), we shall solve in radians ($0 \le x < 2\pi$). Furthermore, once the trigonometric function is isolated, the resulting function value will not appear in the table of exact values. Therefore the corresponding angles will be determined using the inverse trigonometric function keys on the calculator as we have done many times in the past.

5. Solve the following for x in radians ($0 \le x < 2\pi$):
$\tan(3x) + \sec(3x) = 2 \Rightarrow \sec(3x) = 2 - \tan(3x) \Rightarrow (\sec(3x))^2 = (2 - \tan(3x))^2 \Rightarrow$ $\sec^2(3x) = 4 - 4\tan(3x) + \tan^2(3x) \Rightarrow 1 + \tan^2(3x) = 4 - 4\tan(3x) + \tan^2(3x)$ (using the second Pythagorean identity) $\Rightarrow 1 = 4 - 4\tan(3x)$ (subtracting $\tan^2(3x)$ from both sides) $\Rightarrow 4\tan(3x) = 3 \Rightarrow \tan(3x) = \frac{3}{4}$.

But $0 \le x < 2\pi \Rightarrow 0 \le 3x < 6\pi$, so we locate the solutions for $0 \le x < 2\pi$, and then measure these angles counter-clockwise from the positive x-axis, going around the Cartesian Plane <u>3 times</u> since $0 \le 3x < 6\pi$. (See the diagram below.)

Thus $\tan(3x) = \frac{3}{4} \Rightarrow$

$(3x)_{ref} = \tan^{-1}\left(\frac{3}{4}\right) = 0.6435$.

Since tangent is positive in QI, QIII, then
$3x = 0.6435, 3.7851, 6.9267, 10.0683, 13.2099, 16.3515$.

Dividing both sides by 3 yields
$x = 0.2145, 1.2617, 2.3089, 3.3561, 4.4033, 5.4505$,
<u>all of which satisfy $0 \le x < 2\pi$</u>.

However, since we squared both sides of the equation in one step, we may have introduced *extraneous solutions* which satisfied the equation after squaring but not the original equation. Therefore we must check each of these six *potential solutions* in the original equation, bearing in mind that these are only *approximate solutions* from the calculator. Checking each solution in the original equation verifies that 0.2145, 2.3089, 4.4033 are actually solutions, while 1.2617, 3.3561, 5.4505 are extraneous solutions. Hence the actual (approximate) solutions are $x = 0.2145, 2.3089, 4.4033$.