Section 5.6

1. Starting with a Double Angle identity, we have

$$\cos 2A = 1 - 2\sin^2 A \Rightarrow 2\sin^2 A = 1 - \cos 2A \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2} \Rightarrow$$

$$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}.$$ Similarly, $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$, and

$$\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}} = \frac{\sin 2A}{1+\cos 2A} = \frac{1-\cos 2A}{\sin 2A}.$$

These are the Half Angle Identities, so called because the term isolated on the left side of the equation has an angle that is half the angle $2A$ which appears on the right side of the equation.
Note: There are three Half Angle identities for $\tan A$.

2. The Half Angle Identities come in an alternate form. Substituting $\frac{A}{2}$ for A in both sides of these identities, we have

Half Angle Identities

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \quad , \quad \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

3. <u>Note</u>: The $\pm$ sign in the Half Angle Identities depends on the quadrant in which the angle $A/2$ terminates.

4. <u>Example</u>: Compute an <u>exact</u> value of $\sin 165^\circ$ using a Half Angle identity.

$$\sin 165^\circ = \sin\left(\frac{330^\circ}{2}\right) = \pm\sqrt{\frac{1-\cos 330^\circ}{2}}.$$

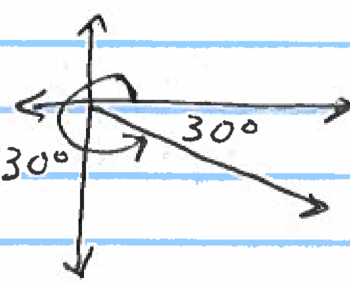
Before proceeding, we resolve the $\pm$ sign first. Since 165° is in QII, then $\sin 165^\circ > 0$, and so

$$\sin 165^\circ = +\sqrt{\frac{1-\cos 330^\circ}{2}} = \sqrt{\tfrac{1}{2}\cdot(1-\cos 330^\circ)} =$$

$$\sqrt{\tfrac{1}{2}\cdot\left(1-\tfrac{\sqrt{3}}{2}\right)} \quad \left(\text{since } \cos 330^\circ = \tfrac{\sqrt{3}}{2}\right) =$$

$$\sqrt{\tfrac{1}{2}\cdot\frac{2-\sqrt{3}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

<u>Note</u>: $\cos 330^\circ = \frac{\sqrt{3}}{2}$ was obtained using the 30° reference angle in QIV, 330° the Table of Exact Values, and the Table of Signs of the trig functions in the four quadrants as we have done before.



5. <u>Example</u>: Compute an exact value of $\cos 195°$ using a Half Angle identity.

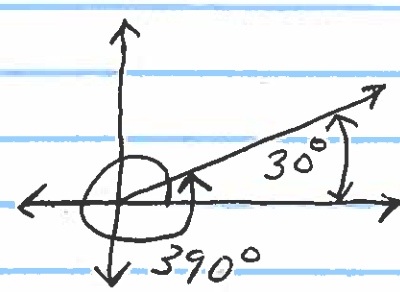
$$\cos 195° = \cos\left(\frac{390°}{2}\right) = \pm\sqrt{\frac{1+\cos 390°}{2}}.$$

To resolve the $\pm$ sign, note that $195°$ is in QIII, so that $\cos 195° < 0$. Therefore

$$\cos 195° = -\sqrt{\frac{1+\cos 390°}{2}} = -\sqrt{\frac{1}{2}\cdot(1+\cos 390°)} =$$

$$-\sqrt{\frac{1}{2}\cdot\left(1+\frac{\sqrt{3}}{2}\right)} = -\sqrt{\frac{1}{2}\cdot\frac{2+\sqrt{3}}{2}} = -\sqrt{\frac{2+\sqrt{3}}{4}} = -\frac{\sqrt{2+\sqrt{3}}}{2}.$$

<u>Note</u>: $\cos 390°$ was obtained using the $30°$ reference angle in QI, the Table of Exact Values, and the Table of Signs of the trig functions in the four quadrants as we have done before.



6. The process illustrated in the preceding examples must be performed in radian mode as well.

7. _Example_: Compute an _exact_ value of $\tan\left(-\frac{\pi}{8}\right)$ using a Half Angle identity.

Note: There are three identities for $\tan\frac{A}{2}$. The first one should virtually _never_ be used, and the last one has advantages over the second one.

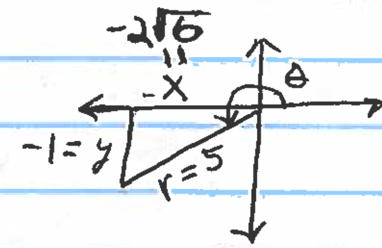
Then $\tan\left(-\frac{\pi}{8}\right) = -\tan\frac{\pi}{8}$ (Negative Angle Identity) $=$

$$-\tan\frac{\left(\frac{\pi}{4}\right)}{2} = -\frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = -\frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\cdot\frac{2}{2} = -\frac{2-\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} =$$

$$-\frac{2\sqrt{2}-2}{2} = -\frac{2\cdot(\sqrt{2}-1)}{2} = -\left(\sqrt{2}-1\right) = -\sqrt{2}+1 = 1-\sqrt{2}.$$

Note: $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ were obtained from the Table of Exact Values (in radians).

8. <u>Example</u>: Compute an <u>exact</u> value of $\cos\frac{\theta}{2}$ if $\sin\theta = -\frac{1}{5}$ and θ is in QIII.



Drawing θ in QIII, we have $\sin\theta = -\frac{1}{5} = \frac{y}{r}$. Since r is always positive, then $r=5$ and $y=-1$.

Using the Pythagorean Theorem, $x^2+y^2=r^2 \Rightarrow x^2+(-1)^2=(5)^2 \Rightarrow x^2+1=25 \Rightarrow x^2=24 \Rightarrow x=\pm\sqrt{24} \Rightarrow x=\pm 2\sqrt{6}$. Furthermore, the diagram shows that x is in the negative direction in QIII, so $x=-2\sqrt{6}$.

Using the Half Angle identity, $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$.

Since θ is in QIII, then $180° \le \theta \le 270° \Rightarrow$

$\frac{180°}{2} \le \frac{\theta}{2} \le \frac{270°}{2} \Rightarrow 90° \le \frac{\theta}{2} \le 135° \Rightarrow \frac{\theta}{2}$ is in QII $\Rightarrow$ $\cos\frac{\theta}{2} < 0$, so we pick the negative sign for the radical. Also, from the diagram, we have $\cos\theta = \frac{x}{r} = \frac{-2\sqrt{6}}{5}$. Therefore

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\frac{-2\sqrt{6}}{5}}{2}} = -\sqrt{\frac{1}{2}\cdot\left(1-\frac{2\sqrt{6}}{5}\right)} = -\sqrt{\frac{1}{2}-\frac{\sqrt{6}}{5}} = -\sqrt{\frac{5-2\sqrt{6}}{10}}.$$

Hence $\cos\frac{\theta}{2} = -\sqrt{\frac{5-2\sqrt{6}}{10}}$.