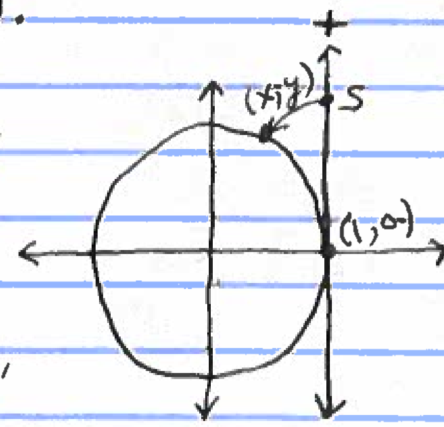
Section 3.3

1. To deal with trig functions in calculus, we must treat them as functions of <u>real numbers</u> rather than <u>angles</u>. The issue then is how to treat an angle like a real number.

2. Recall that when computing trig functions of an angle in standard position, we can use any point on the terminal side of θ due to similar triangles. Therefore we choose to use the point 1 unit from the origin (at the intersection of the terminal side of θ with the unit circle).

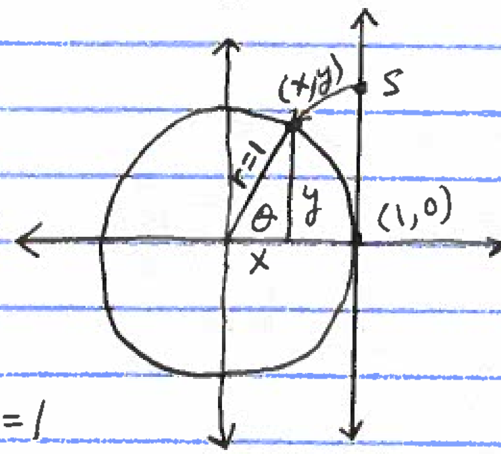


3. In a coordinate system, construct the unit circle (center at $(0,0)$ and radius $=1$) and the vertical line $x=1$ through the point $(1,0)$. Make the line $x=1$ a "number line" by defining the origin 0 at the point $(1,0)$ in the coordinate system, positive direction up, and negative direction down.

On this vertical number line, locate a real number "s". Then wrap that end of the number line around the unit circle until the number "s" falls on the unit circle at a point (x,y) relative to the coordinate system.

Define $\sin(s) = y$, $\cos(s) = x$, $\tan(s) = \frac{y}{x}$,
$\cot(s) = \frac{x}{y}$, $\sec(s) = \frac{1}{x}$, and $\csc(s) = \frac{1}{y}$.

Now construct the ray from $(0,0)$ through (x,y), forming an angle θ relative to the positive x-axis. Note that θ is in standard position. Form the reference Δ with side $x, y, r=1$ by dropping a vertical from (x,y) to $(x,0)$ on the x-axis.



Then $\sin\theta = \frac{y}{r} = \frac{y}{1} = y = \sin(s)$,

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x = \cos(s),$$

$$\tan\theta = \frac{y}{x} = \tan(s), \quad \cot\theta = \frac{x}{y} = \cot(s),$$

$$\sec\theta = \frac{r}{x} = \frac{1}{x} = \sec(s), \quad \csc\theta = \frac{r}{y} = \frac{1}{y} = \csc(s).$$

Thus each trig function of θ equals the same trig function of s. Note also that "s" forms the arclength on the unit circle from $(1,0)$ to (x,y). So relative to θ in <u>radians</u>, we have $\underline{s = r\theta = 1\cdot\theta = \theta}$.

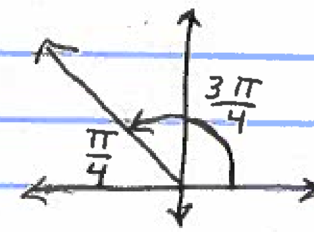
Thus any trig function of a real number "s" is the same as that trig function of the angle $\theta = s$ radians. Hence an angle in radians can be considered a real number.

Table of Exact Trigonometric Values

θ (rad / deg)	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞
$\csc\theta$	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1

4. Examples:

(a) To compute the exact value of the tangent of the real number $\frac{3\pi}{4}$, interpret $\frac{3\pi}{4}$ as an "angle" of $\frac{3\pi}{4}$ radians and proceed as we have done before. Drawing $\frac{3\pi}{4}$ in standard position, we get a reference angle of $\frac{\pi}{4}$. From the Table of Exact Values, $\tan\frac{\pi}{4} = 1$. Finally, $\tan\theta < 0$ in QII. Thus $\tan\frac{3\pi}{4} = -1$.



(b) For real numbers (angles in radians) not in the Table of Exact Values, use the calculator. For the sine of the real number -4.1, simply put the calculator in radian mode and compute $\sin(-4.1) = 0.818277111$.

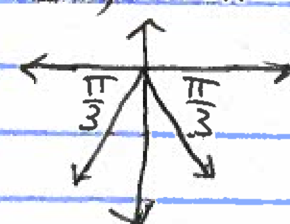
5. Solving trig equations for real number solutions is also performed exactly as before. We simply find the "angle solutions" in radians, which are precisely the real number solutions.

Examples:

(a) Solve $2\sin x + \sqrt{3} = 0$ for real numbers x, $0 \le x < 2\pi$.

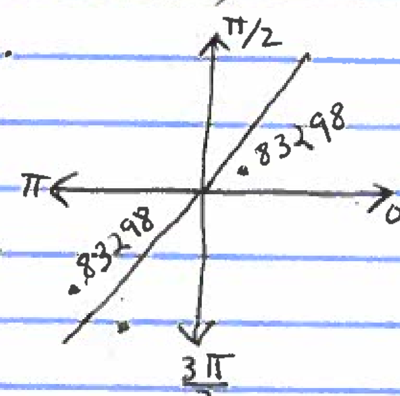
$2\sin x + \sqrt{3} = 0 \Rightarrow 2\sin x = -\sqrt{3} \Rightarrow \sin x = -\frac{\sqrt{3}}{2}$.
Looking across the sine row in the Table of Exact Values, we locate $\frac{\sqrt{3}}{2}$ (ignoring the minus sign). Then moving up to the top row we find $\frac{\pi}{3}$. Thus the reference angle in radians is $x_{ref} = \frac{\pi}{3}$.

Since $-\frac{\sqrt{3}}{2} < 0$ and $\sin\theta < 0$ in QIII & QIV, then draw $x_{ref} = \frac{\pi}{3}$ in QIII & QIV. Measuring these counter-clockwise from the positive x-axis, we get $x = \frac{4\pi}{3}, \frac{5\pi}{3}$, which are the solutions sought.



(b) Solve $2\tan x - 2.2 = 0$ for real numbers x, $0 \le x < 2\pi$.

$2\tan x - 2.2 = 0 \Rightarrow 2\tan x = 2.2 \Rightarrow \tan x = \frac{2.2}{2} = 1.1$.
Then $x_{ref} = \tan^{-1}(1.1) \approx .83298$ (rounded).
Since $\tan x = 1.1 > 0$ and $\tan\theta > 0$ in QI & QIII, then draw $x_{ref} = .83298$ in QI & QIII. Measuring these counter-clockwise from the positive x-axis, we have $x = .83298$ and $x = \pi + .83298 = 3.97457$.



Thus $x = .83298, 3.97457$ are the (approximate) solutions for $0 \le x < 2\pi$.

(c) Solve $3\sec x + 6.9 = 0$ for real numbers x, $0 \le x < 2\pi$.

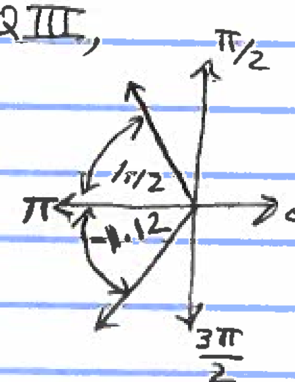
$3\sec x + 6.9 = 0 \Rightarrow 3\sec x = -6.9 \Rightarrow \sec x = \frac{-6.9}{3} \Rightarrow$

$\sec x = -2.3 \Rightarrow \cos x = \frac{1}{\sec x} = -\frac{1}{2.3}$.

Ignoring the minus sign, $x_{ref} = \cos^{-1}\left(\frac{1}{2.3}\right) \approx 1.12$.

Since $\cos x = -\frac{1}{2.3} < 0$ and $\cos\theta < 0$ in QII & QIII, then draw $x_{ref} = 1.12$ rad in QII & QIII. Measuring these counter-clockwise from the positive x-axis, we have

$x = \pi - 1.12 \approx 2.02$ and $x = \pi + 1.12 \approx 4.26$.



Thus $x = 2.02, 4.26$ are the (approximate) solutions sought.

Note: In determing the quadrants in which to draw x_{ref}, we used the fact that $\cos x = -\frac{1}{2.3} < 0$ and $\cos\theta < 0$ in QII & QIII. We could also have used $\sec x = -2.3$ and $\sec\theta < 0$ in QII & QIII. Either approach is valid since $\cos\theta$ and $\sec\theta$ are reciprocals, and reciprocals always have the same sign (both positive or both negative).