Section 5.2

1. An identity is formally an equation which is valid for all acceptable values of the variable(s) involved. When verifying trig identities, do not start with the identity and altar both sides until they are identical. Instead, start with the more complex side only, and transform it into the other side using previously established identities and the rules of ordinary algebra.

The following list provides a list of six hints, or suggestions, of commonly used techniques to verify trig identities. In the examples that follow, it will be pointed out when each of these hints are employed. We now review these six hints.

A list of the fundamental identities established up to now is also provided.

2.

Hints for Verifying Identities & Fundamental Identities

Verifying Identities

1. Learn the fundamental identities given in the last section. Whenever you see either side of a fundamental identity, the other side should come to mind. Also, be aware of equivalent forms of the fundamental identities. For example $\sin^2\theta = 1 - \cos^2\theta$ is an alternative form of $\sin^2\theta + \cos^2\theta = 1$.
2. Try to rewrite the more complicated side of the equation so that it is identical to the simpler side.
3. It is often helpful to express all trigonometric functions in the equation in terms of sine and cosine and then simplify the result.
4. Usually any factoring or indicated algebraic operations should be performed. For example, the expression $\sin^2 x + 2\sin x + 1$ can be factored as follows: $(\sin x + 1)^2$. The sum or difference of two trigonometric expressions, such as

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta},$$

can be added or subtracted in the same way as any other rational expressions:

$$\frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \frac{\cos\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos\theta} = \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta}.$$

5. As you select substitutions, keep in mind the side you are not changing, because it represents your goal. For example, to verify the identity

$$\tan^2 x + 1 = \frac{1}{\cos^2 x},$$

try to think of an identity that relates $\tan x$ to $\cos x$. Here, since $\sec x = 1/\cos x$ and $\sec^2 x = \tan^2 x + 1$, the secant function is the best link between the two sides.
6. If an expression contains $1 + \sin x$, multiplying both numerator and denominator by $1 - \sin x$ would give $1 - \sin^2 x$, which could be replaced with $\cos^2 x$. Similar results for $1 - \sin x$, $1 + \cos x$, and $1 - \cos x$ may be useful.

These hints are used in the examples of this section.

Reciprocal Identities

$$\sin A = \frac{1}{\csc A}, \quad \csc A = \frac{1}{\sin A}$$

$$\cos A = \frac{1}{\sec A}, \quad \sec A = \frac{1}{\cos A}$$

$$\tan A = \frac{1}{\cot A}, \quad \cot A = \frac{1}{\tan A}$$

Quotient Identities

$$\tan A = \frac{\sin A}{\cos A}, \quad \cot A = \frac{\cos A}{\sin A}$$

Negative Angle Identities

$$\sin(-A) = -\sin A, \quad \csc(-A) = -\csc A$$

$$\cos(-A) = \cos A, \quad \sec(-A) = \sec A$$

$$\tan(-A) = -\tan A, \quad \cot(-A) = -\cot A$$

Pythagorean Identities

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$1 + \cot^2 A = \csc^2 A$$

3. The rest of this section is devoted to verifying (non-fundamental) identities using the hints and fundamental identities just presented.

4. <u>Example</u>: Verify $\cot x + 1 = \csc x \cdot (\cos x + \sin x)$.

Since the right side of the equation seems to be more complex, we start with <u>only that side</u> and transform it to the left side.

$\csc x \cdot (\cos x + \sin x) = \csc x \cdot \cos x + \csc x \cdot \sin x =$
(hint #4) (normal rules of algebra)

$\frac{1}{\sin x} \cdot \cos x + \frac{1}{\sin x} \cdot \sin x$ (hint #3) (hint #1; reciprocal identities) $=$

$\frac{\cos x}{\sin x} + 1$ (hint #4; normal rules of algebra) $=$

$\cot x + 1$ (hint #1; quotient identities).

Hence $\csc x \cdot (\cos x + \sin x) = \cot x + 1$.

5. *Example*: Verify $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \tan^2\theta - \cot^2\theta$.

Starting with only the more complex left side of the equation, we have

$\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \frac{\tan\theta}{\sin\theta\cos\theta} - \frac{\cot\theta}{\sin\theta\cos\theta}$ (hint 5; 2 terms on right) (hint 4; rules of algebra) =

$\tan\theta \cdot \frac{1}{\sin\theta\cos\theta} - \cot\theta \cdot \frac{1}{\sin\theta\cos\theta}$ (hint 4; rules of algebra) =

$\frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sin\theta\cos\theta} - \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta\cos\theta}$ (hint 3; convert to $\sin\theta$ & $\cos\theta$) (hint 1; quotient identities) =

$\frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$ (hint 4; rules of algebra) =

$\sec^2\theta - \csc^2\theta$ (hint 1; reciprocal identities) =

$(1+\tan^2\theta) - (1+\cot^2\theta)$ (hint 1; Pythagorean identities) =

$1+\tan^2\theta - 1 - \cot^2\theta$ (hint 4; rules of algebra) =

$\tan^2\theta - \cot^2\theta$ (hint 4; rules of algebra).

Hence $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \tan^2\theta - \cot^2\theta$.

6. To be more specific about hint 6, when the denominator of a fraction contains $1+\sin x$, multiplying by $\frac{1-\sin x}{1-\sin x}$ can be potentially useful.

The same process may be useful if $1+\sin x$ appears in the numerator of a fraction <u>OR</u> as an isolated term not in a fraction.

The same logic applies to $1-\sin x$, $1+\cos x$, & $1-\cos x$ and multiplying by

$\frac{1+\sin x}{1+\sin x}$, $\frac{1-\cos x}{1-\cos x}$, and $\frac{1+\cos x}{1+\cos x}$, respectively.

7. <u>Example</u>: Verify $\frac{\cos x}{1-\sin x} = \frac{1+\sin x}{\cos x}$.

Both sides seem equally complex, so using the left side,

we have $\frac{\cos x}{1-\sin x} = \frac{\cos x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x}$ (hint 6) =

$\frac{\cos x \cdot (1+\sin x)}{1-\sin^2 x}$ (hint 4, rules of algebra) =

$\frac{\cos x \cdot (1+\sin x)}{\cos^2 x}$ (hint 1; altered form of a Pythagorean identity) =

$\frac{\cos x \cdot (1+\sin x)}{\cos x \cdot \cos x}$ (hint 4; rules of algebra) =

$\frac{1+\sin x}{\cos x}$ (hint 4; rules of algebra).

Hence $\frac{\cos x}{1-\sin x} = \frac{1+\sin x}{\cos x}$.

8. Sometimes both sides of an identity can be sufficiently complex so that transforming either side to the other is difficult. In simple terms, the difficulty arises because after starting with one side and simplifying it, we need to "complicate" the simplified version into the relatively complex "other side". It seems that the human mind "simplifies" better than it "complicates" due to there being fewer ways to simplify than to complicate.

In this case we may start with one side and simplify it as much as possible. Then we transform the <u>other side</u> to this <u>simplified version of the first side</u>. What is accomplised is showing that both sides of the original equation are equal to the same simplified expression. Thus the two sides are equal to each other, and the identity is verified.

This may <u>seem</u> like the forbidden process of changing both sides until they are identical, but there is a crucial difference. This process assumes only that two things equal to the same thing are equal to each other. Starting with the original identity assumes the two sides are <u>already equal</u>, which is what needs to be verified!

8. <u>Example</u>: Verify $\tan^2\theta \cdot (1+\cot^2\theta) = \frac{1}{1-\sin^2\theta}$.

Starting with the left side of the equation,

$\tan^2\theta \cdot (1+\cot^2\theta) =$

$\tan^2\theta + \tan^2\theta \cdot \cot^2\theta$ (hint 4; rules of algebra) =

$\tan^2\theta + \tan^2\theta \cdot \frac{1}{\tan^2\theta}$ (hint 1; reciprocal identities) =

$\tan^2\theta + 1$ (hint 4; rules of algebra) =

$\sec^2\theta$ (hint 1, Pythagorean identities).

Simplify the right side of the equation, we have

$\frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$ (hint 1; altered form of Pythagorean identities) =

$\sec^2\theta$ (hint 1; reciprocal identities).

Thus since $\tan^2\theta \cdot (1+\cot^2\theta) = \sec^2\theta$, and

$\frac{1}{1-\sin^2\theta} = \sec^2\theta$, then $\tan^2\theta \cdot (1+\cot^2\theta) = \frac{1}{1-\sin^2\theta}$.

10. *Example*: Verify $\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x}$.

First $\dfrac{\sec x + \tan x}{\sec x - \tan x} =$

$$\frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} \quad \left(\begin{array}{l}\text{hint 3; convert to } \sin x \text{ \& } \cos x \\ \text{hint 1; reciprocal \& quotient identities}\end{array}\right) =$$

$$\frac{\frac{1 + \sin x}{\cos x}}{\frac{1 - \sin x}{\cos x}} \quad (\text{hint 4; rules of algebra}) =$$

$$\frac{1 + \sin x}{\cos x} \cdot \frac{\cos x}{1 - \sin x} \quad (\text{hint 4; rules of algebra}) =$$

$$\frac{1 + \sin x}{1 - \sin x} \quad (\text{hint 4; rules of algebra}).$$

Then $\dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x} =$

$$\frac{(1 + \sin x)^2}{1 - \sin^2 x} \quad \left(\begin{array}{l}\text{hint 4; rules of algebra} \\ \text{hint 1; Pythagorean identities}\end{array}\right) =$$

$$\frac{(1 + \sin x)(1 + \sin x)}{(1 + \sin x)(1 - \sin x)} \quad (\text{hint 4; rules of algebra}) =$$

$$\frac{1 + \sin x}{1 - \sin x} \quad (\text{hint 4; rules of algebra}).$$

Thus since $\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{1 + \sin x}{1 - \sin x}$ and

$\dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x} = \dfrac{1 + \sin x}{1 - \sin x}$, then

$\dfrac{\sec x + \tan x}{\sec x - \tan x} = \dfrac{1 + 2\sin x + \sin^2 x}{\cos^2 x}$, and we are done.