Section 5.5

1. Note that $\sin(2A) = \sin(A+A) = \sin A \cos A + \cos A \sin A$ (using the Sum/Diff identities) $= \sin A \cos A + \sin A \cos A = 2\sin A \cos A$. Thus the "Double Angle" identity for the sine function is $\sin(2A) = 2\sin A \cos A$.

Double Angle identities for $\cos(2A)$ and $\tan(2A)$ are also provided below. Note that there are actually three Double Angles identities for $\cos(2A)$. The 2nd and 3rd identity for $\cos(2A)$ are derived by starting with $\cos 2A = \cos^2 A - \sin^2 A$ and either replacing $\sin^2 A$ with $1-\cos^2 A$ or replacing $\cos^2 A$ with $1-\sin^2 A$, respectively (using the Pythagoream identities).

Double Angle Identities

$$\sin 2A = 2 \cdot \sin A \cdot \cos A$$

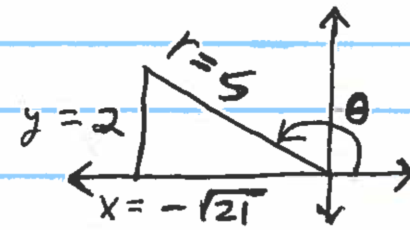
$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cdot \cos^2 A - 1 = 1 - 2 \cdot \sin^2 A$$

$$\tan 2A = \frac{2 \cdot \tan A}{1 - \tan^2 A}$$

2. We now demonstrate how to use the Double Angle identities to compute exact trig values for 2θ given information about θ.

3. Given $\tan\theta = -\frac{2}{\sqrt{21}}$ and $\cos\theta < 0$, compute an <u>exact</u> value for $\sin 2\theta$.

Since $\tan\theta = -\frac{2}{\sqrt{21}} < 0$ and $\cos\theta < 0$, then θ is in QII. Furthermore,



$\tan\theta = -\frac{2}{\sqrt{21}} = \frac{\text{opp}}{\text{adj}} = \frac{y\text{-side}}{x\text{-side}}$. The diagram shows

that the x-side is in the negative direction and the y-side is in the positive direction, so label the x-side $-\sqrt{21}$ and the y-side 2. Then using the Pythagorean Theorem, the hypotenuse is $r = \sqrt{x^2+y^2} = \sqrt{(-\sqrt{21})^2+(2)^2} = \sqrt{21+4} = \sqrt{25} = 5$.

Using the Double Angle identity for $\sin 2\theta$,

we have $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{y}{r}\cdot\frac{x}{r} =$

$2\cdot\frac{2}{5}\cdot\left(\frac{-\sqrt{21}}{5}\right) = \frac{-4\sqrt{21}}{25} = -\frac{4\sqrt{21}}{5}$.

4. We now examine a problem that is essentially the reverse of the preceding problem. That is, we compute exact values of trig functions for θ given information about 2θ.

5. Given $\cos 2\theta = \frac{4}{5}$ and 2θ is in QIV, compute exact values for all trig functions of θ.

Since 2θ is in QIV, then $270° \leq 2\theta \leq 360° \Rightarrow$

$\frac{270°}{2} \leq \frac{2\theta}{2} \leq \frac{360°}{2} \Rightarrow 135° \leq \theta \leq 180° \Rightarrow \theta$ is in QII.

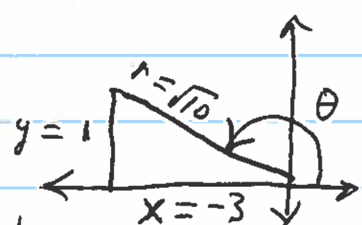
Furthermore, using the identity $\cos 2\theta = 1 - 2\sin^2\theta$,

then $\cos 2\theta = \frac{4}{5} \Rightarrow 1 - 2\sin^2\theta = \frac{4}{5} \Rightarrow 2\sin^2\theta = 1 - \frac{4}{5} \Rightarrow$

$2\sin^2\theta = \frac{1}{5} \Rightarrow \sin^2\theta = \frac{1}{10} \Rightarrow \sin\theta = \pm\sqrt{\frac{1}{10}} = \pm\frac{1}{\sqrt{10}}$.

Since θ is in QII, then $\sin\theta > 0$, and so $\sin\theta = \frac{1}{\sqrt{10}} = \frac{y}{r}$.

Drawing θ in QII, we label the y-side 1 & the r-side $\sqrt{10}$. Then $x^2 + y^2 = r^2 \Rightarrow x^2 + (1)^2 = (\sqrt{10})^2 \Rightarrow$ $x^2 + 1 = 10 \Rightarrow x^2 = 9 \Rightarrow x = \pm\sqrt{9} = \pm 3$. Since the x-side is in the negative direction, then $x = -3$.



Thus $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{10}}$, $\cos\theta = \frac{x}{r} = -\frac{3}{\sqrt{10}}$, $\tan\theta = \frac{y}{x} = -\frac{1}{3}$,

$\cot\theta = \frac{x}{y} = -\frac{3}{1} = -3$, $\sec\theta = \frac{r}{x} = -\frac{\sqrt{10}}{3}$, $\csc\theta = \frac{r}{y} = \frac{\sqrt{10}}{1} = \sqrt{10}$.

6. Simplify $\frac{18\tan(7x)}{3-3\tan^2(7x)}$ using Double Angle identities.

Clearly the given expression looks most like the right side of the Double Angle identity

$\tan 2A = \frac{2\cdot\tan A}{1-\tan^2 A}$. However, the coefficients

in the numerator and denominator should be "2" and "1" rather than "18" an "3". Some simple factoring produces

$$\frac{18\tan(7x)}{3-3\tan^2(7x)} = \frac{9\cdot 2\tan(7x)}{3\cdot(1-\tan^2(7x))} = \frac{9}{3}\cdot\frac{2\tan(7x)}{1-\tan^2(7x)} =$$

$3\cdot\tan(2\cdot 7x)$ (using the Double Angle identity) $=$

$3\cdot\tan(14x)$.

7. Compute an exact value for $24\cos^2(15°) - 12$ using a Double Angle identity.

The expression $24\cos^2(15°) - 12$ looks most like the right side of the Double Angle identity $\cos 2A = 2\cos^2 A - 1$. However, the coefficients of the 1st and 2nd term should be "2" and "1" rather than "24" and "12", respectively. Some simple factoring produces

$24\cos^2(15°) - 12 = 12 \cdot \left[2\cos^2(15°) - 1\right] = 12\cos(30°)$

(using the Double Angle identity) =

$12 \cdot \frac{\sqrt{3}}{2}$ (using the Table of Exact Values) $= 6\sqrt{3}$.

8. Compute an exact value of $\sin 22.5^\circ \cos 22.5^\circ$ using a Double Angle identity.

The expression $\sin 22.5^\circ \cos 22.5^\circ$ looks most like the right side of the Double Angle identity $\sin 2A = 2\sin A \cos A$. However, the factor of "2" is missing from the expression which has a (hidden) factor of 1. A clever trick of replacing "1" with "$\frac{1}{2} \cdot 2$" produces

$\sin 22.5^\circ \cos 22.5^\circ = 1 \cdot \sin 22.5^\circ \cos 22.5^\circ =$

$\left(\frac{1}{2} \cdot 2\right) \sin 22.5^\circ \cos 22.5^\circ = \frac{1}{2} \cdot \left[2 \sin 22.5^\circ \cos 22.5^\circ\right] =$

$\frac{1}{2} \cdot \sin(2 \cdot 22.5^\circ)$ (using $2\sin A \cos A = \sin 2A$) $=$

$\frac{1}{2} \cdot \sin(45^\circ) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$ (Table of Exact Values) $= \frac{\sqrt{2}}{4}$.