Section 8.4

1. Exponents in trigonometric form.

De Moivre's Theorem: If n is a positive integer, then

$$[r\cdot(\cos\theta + i\cdot\sin\theta)]^n = r^n\cdot(\cos(n\theta)+i\cdot\sin(n\theta)).$$

That is, simply exponentiate the magnitude r by a power of n and multiply the angle by n.

Note: If $n\theta \geq 360°$ (or $n\theta \geq 2\pi$), then reduce $n\theta$ to a coterminal angle between $0°$ and $360°$ (or between 0 and 2π) by subtracting $360°$ (or subtracting 2π) as many times as necessary to get an angle between $0°$ and $360°$ (or between 0 and 2π (radians)).

2. Examples:

(a) $[5\cdot(\cos 110° + i\cdot\sin 110°)]^3 = 5^3\cdot(\cos(3\cdot 110°)+i\cdot\sin(3\cdot 110°)) =$
$125\cdot(\cos 330° + i\cdot\sin 330°)$, where $0° \leq 330° < 360°$.

(b) $[6\cdot(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3})]^4 = 6^4\cdot(\cos(4\cdot\frac{\pi}{3})+i\cdot\sin(4\cdot\frac{\pi}{3})) =$

$1296(\cos\frac{4\pi}{3} + i\cdot\sin\frac{4\pi}{3})$, where $0 \leq \frac{4\pi}{3} < 2\pi$.

(c) $\left[2\cdot(\cos 170^\circ + i\cdot\sin 170^\circ)\right]^8 = 2^8\cdot\left(\cos(8\cdot 170^\circ) + i\cdot\sin(8\cdot 170^\circ)\right) =$

$256\cdot(\cos 1360^\circ + i\cdot\sin 1360^\circ) =$

$256\cdot(\cos 280^\circ + i\cdot\sin 280^\circ)$ (by subtracting 360° from 1360° <u>3 times</u>),

where $0^\circ \le 280^\circ < 360^\circ$.

(d) $\left[3\cdot\left(\cos\frac{5\pi}{4} + i\cdot\sin\frac{5\pi}{4}\right)\right]^5 = 3^5\cdot\left(\cos\left(5\cdot\frac{5\pi}{4}\right) + i\cdot\sin\left(5\cdot\frac{5\pi}{4}\right)\right) =$

$243\cdot\left(\cos\frac{25\pi}{4} + i\cdot\sin\frac{25\pi}{4}\right)$

But $2\pi = \frac{8\pi}{4}$, so $\frac{25\pi}{4} > 2\pi$.

Thus $\frac{25\pi}{4}$ reduces to $\frac{\pi}{4}$ by subtracting $2\pi = \frac{8\pi}{4}$ <u>3 times</u>.

Hence $\left[3\left(\cos\frac{5\pi}{4} + i\cdot\sin\frac{5\pi}{4}\right)\right]^5 =$

$243\cdot\left(\cos\frac{25\pi}{4} + i\cdot\sin\frac{25\pi}{4}\right) = 243\cdot\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right)$,

where $0 \le \frac{\pi}{4} < 2\pi$.

3. Roots of Complex Numbers in Trigonometric Form.

Definition of a root of a complex number:

$$\sqrt[n]{x+yi} = a+bi \text{ if and only if } (a+bi)^n = x+yi.$$

4. Theorem: Every complex number has n n^{th} roots.

Thus each complex number has 2 square roots, 3 cube roots, 4 4^{th} roots, etc.

5. Theorem: If n is a positive integer and r is a positive real number, then the n n^{th} roots of $r\cdot(\cos\theta + i\sin\theta)$ are

$$\sqrt[n]{r\cdot(\cos\theta+i\cdot\sin\theta)} = \sqrt[n]{r}\cdot\left(\cos\alpha + i\cdot\sin\alpha\right),$$

$$\text{where } \alpha = \frac{\theta + 360^\circ\cdot k}{n} \quad (0 \le k \le n-1).$$

Note: $0 \le k \le n-1$ yields n distinct values of k, which in turn yields n distinct values of α, which results in n distinct n^{th} roots of $r\cdot(\cos\theta+i\cdot\sin\theta)$.

Observations: $\alpha = \dfrac{\theta+360^\circ\cdot k}{n} = \dfrac{\theta}{n} + \dfrac{360^\circ}{n}\cdot k$

$k = 0 \Rightarrow \alpha = \dfrac{\theta}{n}$, the "principle n^{th} root".

The other term $\dfrac{360^\circ}{n}\cdot k$ indicates that all n roots are equally spaced $\dfrac{360^\circ}{n}$ apart as vectors in the Complex Plane.

6. Alternate approach to computing the n n^{th} roots of $r\cdot(\cos\theta + i\cdot\sin\theta)$:

(a) Compute $\sqrt[n]{r}\cdot\left(\cos\frac{\theta}{n} + i\cdot\sin\frac{\theta}{n}\right)$, the principle n^{th} root of $r\cdot(\cos\theta + i\cdot\sin\theta)$.

(b) Compute $\frac{360°}{n}$, the angle between consecutive roots as vectors in the Complex Plane.

(c) Repeatedly add $\frac{360°}{n}$ to each root, starting with the principle root, until all n roots are obtained.

7. Note: The magnitudes on all the n^{th} roots of $r\cdot(\cos\theta + i\cdot\sin\theta)$ is $\sqrt[n]{r}$.

8. Note: When computing the n n^{th} roots of $r\cdot(\cos\theta + i\cdot\sin\theta)$, the result should be n distinct n^{th} roots of the form $\sqrt[n]{r}\cdot(\cos\alpha + i\cdot\sin\alpha)$, where $0° \le \alpha < 360°$ or $0 \le \alpha < 2\pi$.

9. After computing the principle n^{th} root $\sqrt[n]{r}\cdot\left(\cos\frac{\theta}{n} + i\cdot\sin\frac{\theta}{n}\right)$, write $n-1$ more expressions of the form $\sqrt[n]{r}\cdot(\cos\_\_\_ + i\cdot\sin\_\_\_)$. Then add $\frac{360°}{n}$ (or $\frac{2\pi}{n}$) to each α to get the next one.

10. Examples:

(a) $\sqrt[5]{3125\cdot(\cos 200° + i\cdot\sin 200°)} =$

principle 5^{th} root

$\sqrt[5]{3125}\cdot\left(\cos\frac{200°}{5} + i\cdot\sin\frac{200°}{5}\right) = 5\cdot(\cos 40° + i\cdot\sin 40°)$

This is the principle 5^{th} root, Since there are 5 5^{th} roots, four more are needed, so we write four more with the same magnitude, leaving the angle blank.

$$\left\{\begin{array}{l} 5\cdot(\cos\_\_ + i\cdot\sin\_\_) \\ 5\cdot(\cos\_\_ + i\cdot\sin\_\_) \\ 5\cdot(\cos\_\_ + i\cdot\sin\_\_) \\ 5\cdot(\cos\_\_ + i\cdot\sin\_\_) \end{array}\right.$$

Compute $\frac{360°}{5} = 72°$, where 5 is the root index of the 5^{th} root. Starting with the principle 5^{th} root, add 72° to each angle to get the next angle.

$$\left\{\begin{array}{l} 5\cdot(\cos 40° + i\cdot\sin 40°) \\ \quad +72° \\ 5\cdot(\cos 112° + i\cdot\sin 112°) \\ \quad +72° \\ 5\cdot(\cos 184° + i\cdot\sin 184°) \\ \quad +72° \\ 5(\cos 256° + i\cdot\sin 256°) \\ \quad +72° \\ 5\cdot(\cos 328° + i\cdot\sin 328°) \end{array}\right.$$

The result is the five 5^{th} roots desired.

Note: We knew to stop with 5 roots. However, if we had mistakenly tried to create another root, the angle would be $328° + 72° = 400° \geq 360°$, which indicates too many roots.

(b) $\sqrt[4]{16(\cos 248° + i\cdot\sin 248°)} =$

Principle 4th root

$$\sqrt[4]{16}\cdot\left(\cos\frac{248°}{4} + i\cdot\sin\frac{248°}{4}\right) = 2\cdot(\cos 62° + i\cdot\sin 62°)$$

$+90°$

$\frac{360°}{4} = 90°$

4th root index

$2\cdot(\cos 152° + i\cdot\sin 152°)$

$+90°$

$2\cdot(\cos 242° + i\cdot\sin 242°)$

$+90°$

$2\cdot(\cos 332° + i\cdot\sin 332°)$

These are the 4 4th roots of $16(\cos 248° + i\cdot\sin 248°)$.

(c) $\sqrt[3]{27\left(\cos\frac{5\pi}{4} + i\cdot\sin\frac{5\pi}{4}\right)} =$

Principle cube root

$$\sqrt[3]{27}\cdot\left(\cos\left(\frac{\frac{5\pi}{4}}{3}\right) + i\cdot\sin\left(\frac{\frac{5\pi}{4}}{3}\right)\right) = 3\cdot\left(\cos\frac{5\pi}{12} + i\cdot\sin\frac{5\pi}{12}\right)$$

$+\frac{2\pi}{3} = \frac{8\pi}{12}$

$\frac{2\pi}{3} = \frac{1 \text{ full rotation}}{\text{cube root index}}$

$3\cdot\left(\cos\frac{13\pi}{12} + i\cdot\sin\frac{13\pi}{12}\right)$

$+\frac{2\pi}{3} = \frac{8\pi}{12}$

$3\cdot\left(\cos\frac{21\pi}{12} + i\cdot\sin\frac{21\pi}{12}\right)$

These are the three cube roots of $27\left(\cos\frac{5\pi}{4} + i\cdot\sin\frac{5\pi}{4}\right)$.

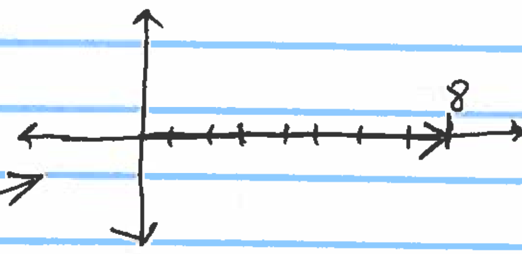
Note: We know to stop with 3 cube roots, but if we had mistakenly tried to create another root, the angle would be $\frac{21\pi}{12} + \frac{2\pi}{3} = \frac{29\pi}{12} \geq 2\pi = \frac{24\pi}{12}$, which indicates too many roots.

11. Perhaps you have seen someone "solve" an equation as follows:

$$x^3 - 8 = 0 \Rightarrow x^3 = 8 \Rightarrow x = \sqrt[3]{8} \Rightarrow x = 2.$$

They were doing pretty well until the last step! $x = \sqrt[3]{8}$ is correct, but we know now that there are <u>three</u> cube roots of 8, not just <u>one</u>.

Drawing "8" as a vector in the Complex Plane, we have the diagram here. Clearly the magnitude (length) is 8, and the angle measured from the positive real (horizontal) axis is $0°$.



Thus $x = \sqrt[3]{8} = \sqrt[3]{8(\cos 0° + i\cdot\sin 0°)}$

principle cube root

$= \sqrt[3]{8}\left(\cos\frac{0°}{3} + i\cdot\sin\frac{0°}{3}\right) = 2(\cos 0° + i\cdot\sin 0°)$

+120°

$\dfrac{1 \text{ full rotation}}{\text{cube root index}} = \dfrac{360°}{3} = 120°$ $\quad 2(\cos 120° + i\cdot\sin 120°)$

+120°

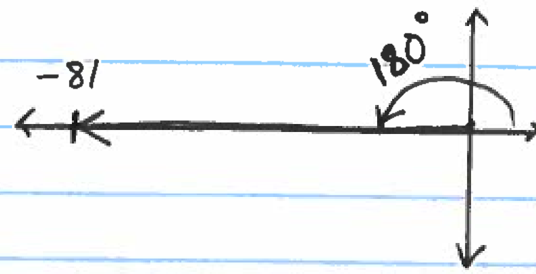
$2(\cos 240° + i\cdot\sin 240°)$

These are the <u>3</u> solutions of $x^3 - 8 = 0$.

Note that $\cos 0° = 1$ and $\sin 0° = 0$, so the principal cube root $2(\cos 0° + i\cdot\sin 0°) = 2(1+0) = 2$.

Thus $x = 2$ is a solution of $x^3 - 8 = 0$, but only a <u>partial</u> solution. We know the "rest of the story"!

12. Drawing -81 as a vector in the Complex Plane, it is clear that the vector has magnitude (length) 81 and direction angle $180°$.



Therefore $x^4 + 81 = 0 \Rightarrow x^4 = -81 \Rightarrow$

$$x = \sqrt[4]{-81} \Rightarrow x = \sqrt[4]{81(\cos 180° + i \cdot \sin 180°)} \Rightarrow$$

$$x = \sqrt[4]{81} \cdot \left(\cos \frac{180°}{4} + i \cdot \sin \frac{180°}{4}\right) = 3 \cdot (\cos 45° + i \cdot \sin 45°)$$

$$\frac{\text{1 full rotation}}{\text{4th root index}} = \frac{360°}{4} = 90°$$

$+90°$

$3 \cdot (\cos 135° + i \cdot \sin 135°)$

$+90°$

$3(\cos 225° + i \cdot \sin 225°)$

$+90°$

$3(\cos 315° + i \cdot \sin 315°)$

These are the 4 solutions of $x^4 + 81 = 0$.

13. Theorem: A polynomial equation of degree n will have n complex solutions.