Section 6.4

Some equations contain inverse trigonometric functions rather than trigonometric functions. When solving such equations, all solutions must be checked in the original equations.

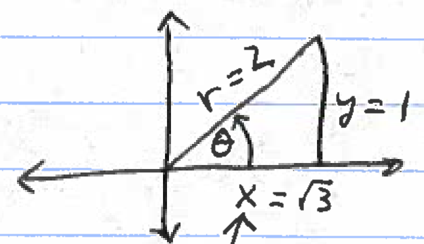
1. Solve the following for x:
$2\cdot\sin^{-1}x - \pi = 0 \Rightarrow 2\cdot\sin^{-1}x = \pi \Rightarrow \sin^{-1}x = \frac{\pi}{2} \Rightarrow$
$x = \sin\frac{\pi}{2} \Rightarrow x = 1.$

Since $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$ and $\frac{\pi}{2}$ is the only angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine value is $+1$, then $\sin^{-1}(1) = \frac{\pi}{2}$. Checking in the original equation, we get $2\cdot\sin^{-1}(1) - \pi = 2\cdot\frac{\pi}{2} - \pi = \pi - \pi = 0$. Thus $x=1$ satisfies the original equations. Since no other solutions were found, then $x=1$ is the only solution.

2. Solve the following for u:

$\cos^{-1} u - \sin^{-1}\left(\frac{1}{2}\right) = 0 \Rightarrow \cos^{-1} u = \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow u = \cos\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$.

Let $\theta = \sin^{-1}\left(\frac{1}{2}\right) \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\sin\theta = \frac{1}{2}$ (positive)
Since sine < 0 in QIV and sine > 0 in QI,
then θ is in QI.
Also, $\sin\theta = \frac{1}{2} = \frac{y}{r}$, so
set $y = 1$ and $r = 2$.



Then $x^2 + y^2 = r^2 \Rightarrow x^2 + (1)^2 = (2)^2 \Rightarrow$
$x^2 + 1 = 4 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$.
Since θ is in QI, then $x > 0$, so $x = +\sqrt{3}$.

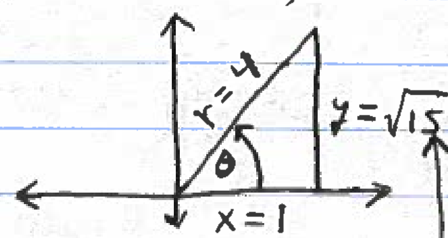
Thus $u = \cos\left[\sin^{-1}\left(\frac{1}{2}\right)\right] = \cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$.
Hence $u = \frac{\sqrt{3}}{2}$ is the solution.

Checking $u = \frac{\sqrt{3}}{2}$ in the original equation
shows that the equation is satisfied.

3. Solve the following for u:

$$\tan^{-1} u - \cos^{-1}\left(\tfrac{1}{4}\right) = 0 \Rightarrow \tan^{-1} u = \cos^{-1}\left(\tfrac{1}{4}\right) \Rightarrow u = \tan\left[\cos^{-1}\left(\tfrac{1}{4}\right)\right].$$

Let $\theta = \cos^{-1}\left(\frac{1}{4}\right) \Rightarrow 0 \le \theta \le \pi$ and $\cos\theta = \frac{1}{4}$ (positive).
Since cosine > 0 in QI and cosine < 0 in QII,
then θ is in QI.
Also, $\cos\theta = \frac{1}{4} = \frac{x}{r}$, so
set $x = 1$ and $r = 4$.



Then $x^2 + y^2 = r^2 \Rightarrow (1)^2 + y^2 = (4)^2 \Rightarrow$
$1 + y^2 = 16 \Rightarrow y^2 = 15 \Rightarrow y = \pm\sqrt{15}$.
Since θ is in QI, then $y > 0$, so $y = +\sqrt{15}$.

Thus $u = \tan\left[\cos^{-1}\left(\frac{1}{4}\right)\right] = \tan\theta = \frac{y}{x} = \frac{\sqrt{15}}{1} = \sqrt{15}$.

Hence $u = \sqrt{15}$ is the solution.

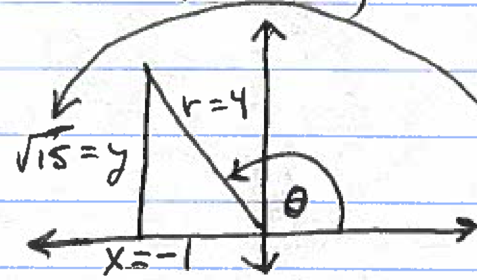
Checking $u = \sqrt{15}$ in the original equation shows that the equation is satisfied.

Compare the next problem to the preceding one.

4. Solve the following for u:

$\tan^{-1}u - \cos^{-1}\left(-\frac{1}{4}\right) = 0 \Rightarrow \tan^{-1}u = \cos^{-1}\left(-\frac{1}{4}\right) \Rightarrow$
$u = \tan\left[\cos^{-1}\left(-\frac{1}{4}\right)\right].$

Let $\theta = \cos^{-1}\left(-\frac{1}{4}\right) \Rightarrow 0 \le \theta \le \pi$ and $\cos\theta = -\frac{1}{4}$ (negative).
Since cosine > 0 in QI and cosine < 0 in QII,
then θ is in QII.
Also, $\cos\theta = -\frac{1}{4} = \frac{x}{r}$, so
set $x = -1$ and $r = 4$.



Note: x gets the minus sign
since x is in the negative direction in QII
and r is always positive.

Then $x^2 + y^2 = r^2 \Rightarrow (-1)^2 + y^2 = (4)^2 \Rightarrow$
$1 + y^2 = 16 \Rightarrow y^2 = 15 \Rightarrow y = \pm\sqrt{15}.$
Since θ is in QII, then $y > 0$, so $y = +\sqrt{15}$.

Thus $u = \tan\left[\cos^{-1}\left(-\frac{1}{4}\right)\right] = \tan\theta = \frac{y}{x} = \frac{\sqrt{15}}{-1} = -\sqrt{15}.$

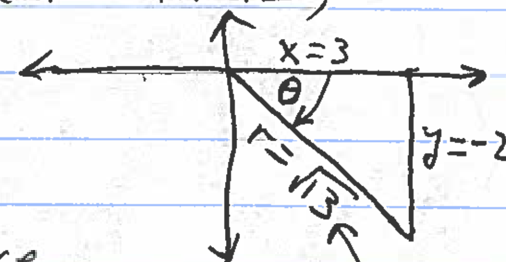
Checking $u = -\sqrt{15}$ in the original equation shows that the equation is NOT satisfied.

Hence there is no solution to the original equation.

5. Solve the following for u:

$\csc^{-1} u - \tan^{-1}\left(-\frac{2}{3}\right) = 0 \Rightarrow \csc^{-1} u = \tan^{-1}\left(-\frac{2}{3}\right) \Rightarrow$
$u = \csc\left[\tan^{-1}\left(-\frac{2}{3}\right)\right]$.

Let $\theta = \tan^{-1}\left(-\frac{2}{3}\right) \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\tan\theta = -\frac{2}{3}$ (negative).
Since tangent > 0 in QI and tangent < 0 in QIV,
then θ is in QIV.



Also, $\tan\theta = -\frac{2}{3} = \frac{y}{x}$, so
set $y = -2$ and $x = 3$.

Note: y gets the minus sign since
x is in the positive direction in QIV and
y is in the negative direction in QIV.

Then $x^2 + y^2 = r^2 \Rightarrow (3)^2 + (-2)^2 = r^2 \Rightarrow 9 + 4 = r^2 \Rightarrow$
$r^2 = 13 \Rightarrow r = \pm\sqrt{13}$.
Since r is always positive, then $r = +\sqrt{13}$.

Thus $u = \csc\left[\tan^{-1}\left(-\frac{2}{3}\right)\right] = \csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{-2} = -\frac{\sqrt{13}}{2}$.

Checking $u = -\frac{\sqrt{13}}{2}$ in the original equation
shows that the equation is satisfied.

Hence $u = -\frac{\sqrt{13}}{2}$ is the solution.