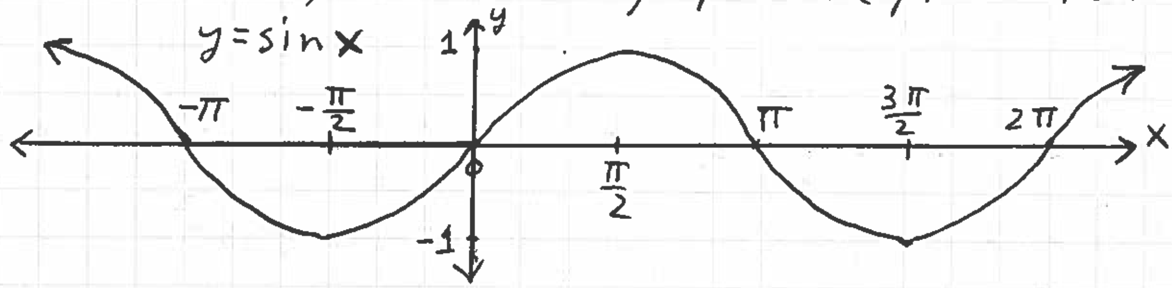
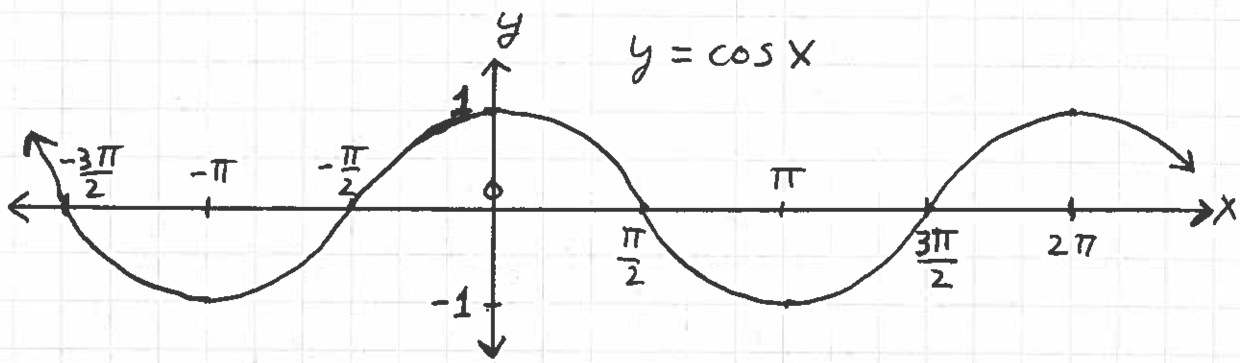
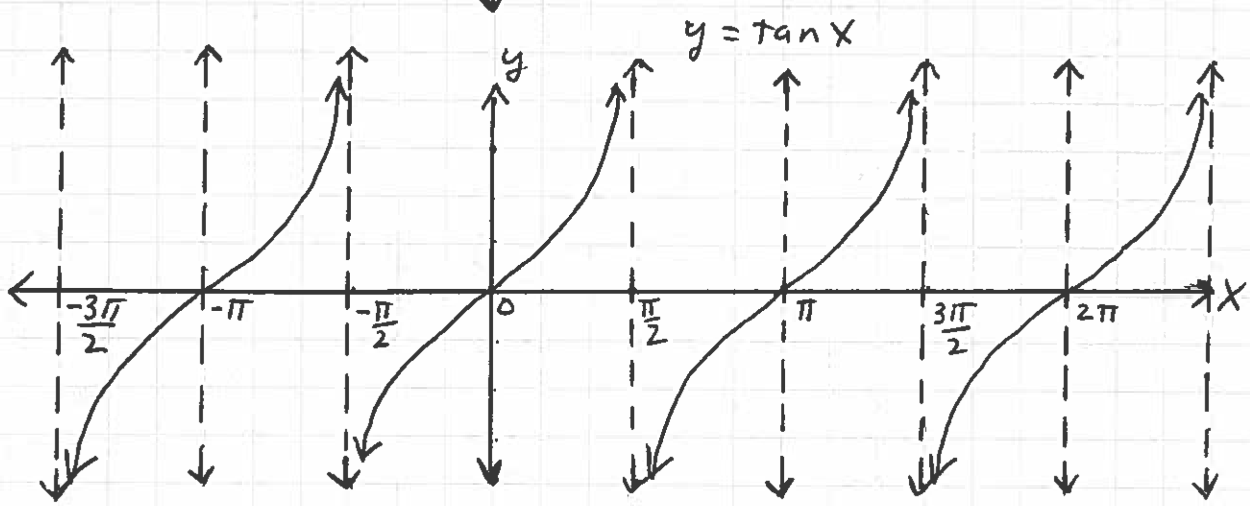
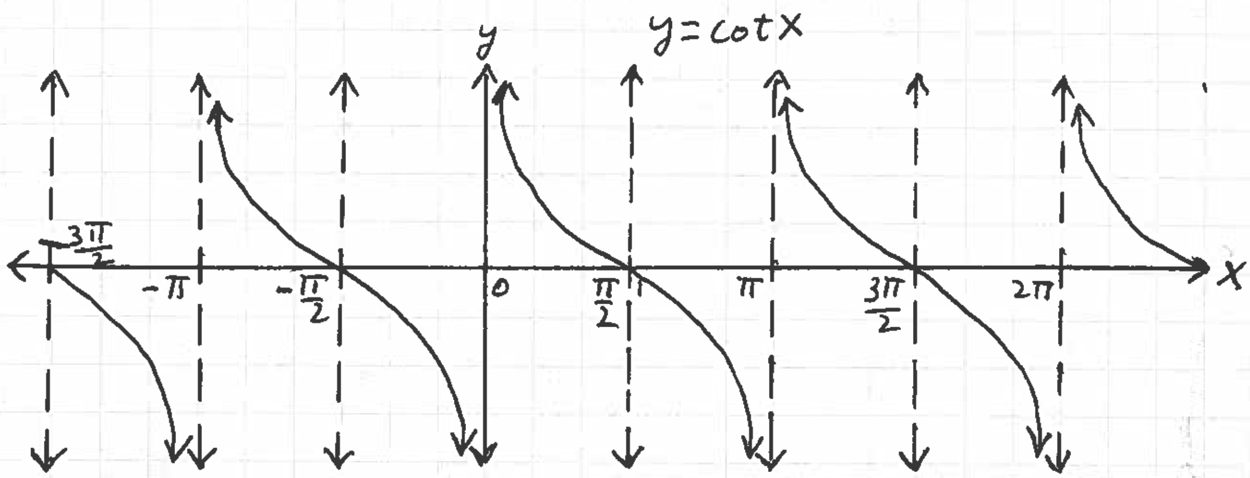
Chapter 4

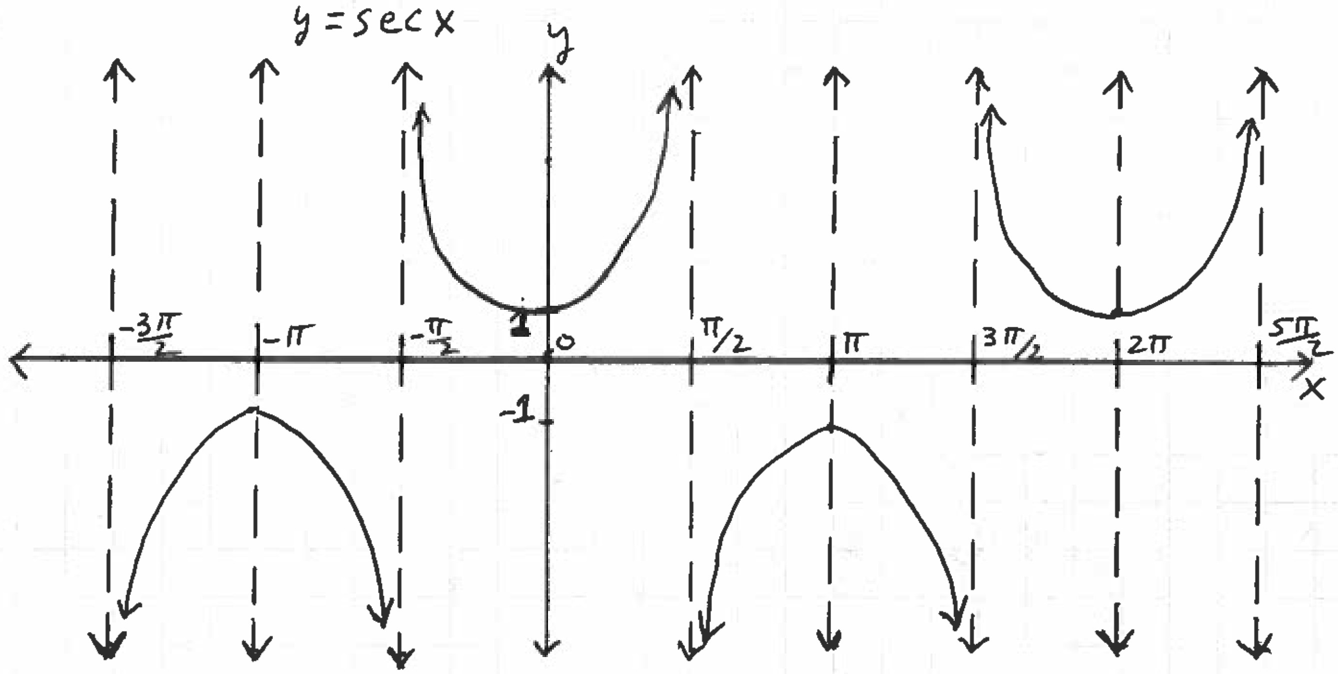
1. Regarding the trig functions as functions of a real number x, their basic graphs are provided below.

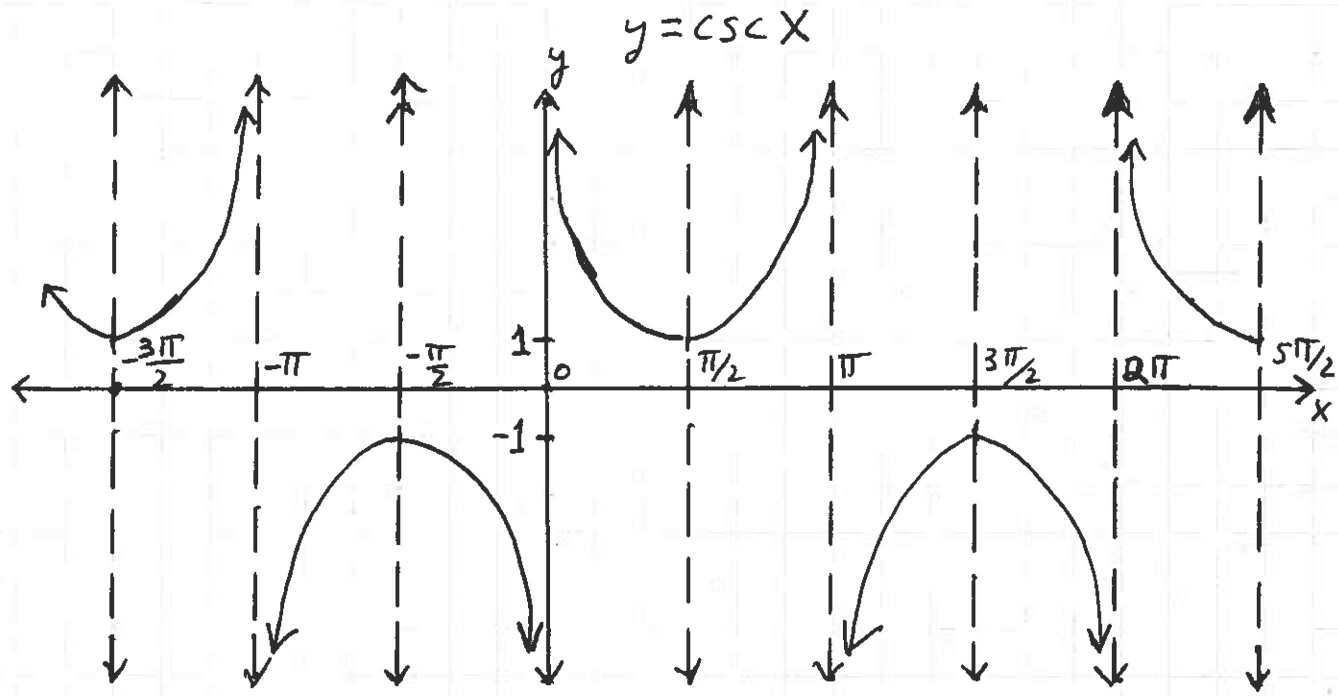









y = sec x
y
x
$-\frac{3\pi}{2}$
$-\pi$
$-\frac{\pi}{2}$
0
$\pi/2$
π
$3\pi/2$
2π
$5\pi/2$
1
-1


y = csc x
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x
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$-\pi$
$-\frac{\pi}{2}$
0
$\pi/2$
π
$3\pi/2$
2π
$5\pi/2$
1
-1

2. All six trig functions are "periodic" in nature, which means their graphs repeat themselves over intervals of equal length in both directions. The lengths of these intervals are called the "period" of the corresponding function.

Examining the graph $y = \sin x$, we see that the part of the graph for $0 \le x \le 2\pi$ is repeated every interval of length 2π. Thus the period of $y = \sin x$ is 2π. Using similar observations for the other trig functions we have the following table for the trig functions and their periods individually.

function	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
period	2π	2π	π	π	2π	2π

3. Note that $y = \sin x$ and $y = \cos x$ both have maximum function values of 1 and minimum function values of -1, unlike the other trig functions. Thus we say that $y = \sin x$ and $y = \cos x$ have "an amplitude of 1", since 1 is the maximum vertical distance these functions obtain from the x-axis. Since the other trig functions have values from $-\infty$ to $+\infty$, they have no amplitudes.

4. The goal of this chapter is to determine the effect on the geometrical graphs of the trig functions when their algebraic formulas are modified in 4 specific ways. More specifically, if a,b,c, & d are real numbers, we want to determine how the graph of $y=a\cdot\sin(bx+c)+d$ compares to $y=\sin x$, and similarly for the other trig functions.

Provided below is a 1-page summary of the effects of all 4 parameters (a,b,c,d) on all 6 trig functions.

A second page is provided below to demonstrate a simple and efficient format in which to answer the homework exercises provided later. Solutions to those exercises are also provided.

Remember to always examine the parameter "b" first. "b" must be positive for the information provided to apply. If $b<0$, then the appropriate negative angle identity must be applied to convert to $b>0$.

Chapter 4 Lecture

The following is a description of the changes to the graphs of the basic trigonometric functions based on the parameters a, b, c, and d in modified functions of the forms: $y = a \cdot \sin(bx + c) + d \ \ (b > 0)$,

$y = a \cdot \cos(bx + c) + d \ \ (b > 0)$,

$y = a \cdot \tan(bx + c) + d \ \ (b > 0)$,

$y = a \cdot \cot(bx + c) + d \ \ (b > 0)$,

$y = a \cdot \sec(bx + c) + d \ \ (b > 0)$, and

$y = a \cdot \csc(bx + c) + d \ \ (b > 0)$

Note: If $b < 0$, first use the appropriate negative angle identity to make $b > 0$.

Parameter "a" $(b > 0)$:

1. The vertical distance of each point from the x-axis is multiplied by $|a|$.

   Note: For sin(x) and cos(x), the amplitude becomes $|a|$.

2. If $a < 0$, the graph is reflected through the x-axis (or revolved about the x-axis).

Parameter "b" $(b > 0)$:

1. The period of the function is $\dfrac{\text{the original period}}{b}$

   Note: The original period of sin, cos, sec, and csc is 2π.
   The original period of tan and cot is π.

Parameter "c" $(b > 0)$:

1. There is a horizontal shift of $-\dfrac{c}{b}$ units.

   Note: The parameter "c" works in conjunction with "b".

   Note: If $-\dfrac{c}{b} < 0$, the shift is to the left.

   If $-\dfrac{c}{b} > 0$, the shift is to the right.

Parameter "d" $(b > 0)$:

1. There is a vertical shift of d units.

   Note: If $d < 0$, the shift is downward.
   If $d > 0$, the shift is upward.

<u>Chapter 4 Homework Template</u>

Parameter "a":

1. The graph is reflected through the x-axis (or revolved about the x-axis).

2. The vertical distance of each point from the x-axis is multiplied by ________.
   OR
   The amplitude changes from _______ to _______ (for sin and cos <u>only</u>).

Parameter "b" (b > 0):

3. The period changes from ________ to _______.

Parameter "c":

4. There is a horizontal shift of _______ units ___________.

Parameter "d":

5. There is a vertical shift of _______ units ___________.

5. We now focus on examples to clarify what is expected in this chapter from the students. With each example provided, examine how the solution matches the format of the Homework Template provided.

6. Note that in $y = a \cdot \sin(bx + c) + d$, the parameters "a" and "b" are multiplied, while "c" and "d" are added. Since numbers are unchanged when multiplied by 1 or added to 0, we will say that "a & b have default values of 1" and that "c & d have default values of 0". Thus if any parameter (a, b, c, or d) has its default value in $y = a \cdot \sin(bx + c) + d$, then that parameter will not appear in the equation, it will have no effect from the basic function $y = \sin x$, and there is <u>no need to mention that parameter</u> in the solution. For reference, the default values for each parameter are provided in the table below. Also, the information provided here for $y = a \cdot \sin(bx + c) + d$ is valid for <u>all</u> the trig functions.

Parameter	a	b	c	d
Default Value	1	1	0	0

For example, $y = \cos x \Rightarrow y = 1 \cdot \cos(1 \cdot x + 0) + 0$, so $a = 1$, $b = 1$, $c = 0$, and $d = 0$.

7. In each of the following examples, state how the graph of the given equation differs from the graph of the corresponding basic trig function. Follow the format of the Homework Template.

8. Example: $y = \frac{2}{3}\sin x \quad \left(= \frac{2}{3}\sin(1 \cdot x + 0) + 0\right)$.

Thus $a = \frac{2}{3}$, $b = 1$, $c = 0$, $d = 0$.

Since $a = \frac{2}{3} > 0$, we need not say that the graph is not reflected through the x-axis relative to $y = \sin x$. However, we do state that the amplitude changes from 1 to $\left|\frac{2}{3}\right| = \frac{2}{3}$, or just state that the amplitude is $\left|\frac{2}{3}\right| = \frac{2}{3}$.

Since all other parameters have their default value, then they have no effect on $y = \sin x$. Thus we do not mention them, and we are done.

9. Example: $y = -3\tan x$ $\left(= -3\tan(1x+0)+0\right)$.

Thus $a=-3$, $b=1$, $c=0$, $d=0$.

Since $a=-3<0$, the graph is reflected through the x-axis (relative to $y=\tan x$).

Also, the vertical distance of each point from the x-axis is multiplied by $|-3|=3$.
Note: Only $\sin x$ and $\cos x$ have an amplitude, so we cannot express the effect of $a=-3$ in terms of amplitude.

All other parameters have their default values, so we are done.

10. Example: $y=\cot(3x)$ $\left(= 1\cdot\cot(3x+0)+0\right)$.

Thus $a=1$, $b=3$, $c=0$, $d=0$.

All parameters have default values except $b=3$, so we consider only the parameter $b=3$.
Since the period of $y=\cot x$ is π (not 2π), then the period of $y=\cot(3x)$ is $\frac{\pi}{3}$.

11. <u>Example</u>: $y=\sec\left(\frac{x}{2}\right)$ $\left(=1\cdot\sec\left(\frac{1}{2}\cdot x+0\right)+0\right)$.

Thus $a=1$, $b=\frac{1}{2}$ (<u>not</u> 2), $c=0$, $d=0$.

Since $a, c, \& d$ have default values, we consider only $b=\frac{1}{2}$. Since $y=\sec x$ has period 2π, then

$y=\sec\left(\frac{x}{2}\right)$ has period $\frac{2\pi}{\frac{1}{2}}=4\pi$ (<u>not</u> π).

The other parameters have no effect, so we are done.

12. Note that the parameter "c" does not work alone, but in conjunction with "b". Thus when $c\neq 0$, then c is used with b to compute $-\frac{c}{b}$

13. <u>Example</u>: $y=\sin\left(x-\frac{\pi}{6}\right)$ $\left(=1\cdot\sin\left(1x+\frac{-\pi}{6}\right)+0\right)$.

Thus $a=1$, $b=1$, $c=-\frac{\pi}{6}$, $d=0$.
Since $a, b, \& d$ have default values, they have no effect.
There is a horizontal shift of $-\frac{c}{b}=-\frac{-\frac{\pi}{6}}{1}=+\frac{\pi}{6}$ units.
Since the horizontal axis (x-axis) is positive to the right & negative to the left, we interpret $-\frac{c}{b}=+\frac{\pi}{6}$ as a "horizontal shift $\frac{\pi}{6}$ units to the right."
Since the other parameters have default values, we are done.

14. Example: $y = \sec(x + \frac{\pi}{4})$ $(= 1 \cdot \sec(1 \cdot x + \frac{\pi}{4}) + 0)$.

Thus $a = 1, b = 1, c = \frac{\pi}{4}, d = 0$.

Since $a, b, \& d$ have default values, they have no effect and we consider only c (in conjunction with b). There is a horizontal shift of

$-\frac{c}{b} = -\frac{\frac{\pi}{4}}{1} = -\frac{\pi}{4}$ units, which we state as a

"horizontal shift of $-\frac{\pi}{4}$ units to the left".

15. Since the vertical axis (y-axis) is positive upward and negative downward, we interpret $d > 0$ as a shift up and $d < 0$ as a shift down.

16. Example: $y = \tan x - 1$ $(= 1 \cdot \tan(1 \cdot x + 0) - 1)$.

Thus $a = 1, b = 1, c = 0, d = -1$.

Since $a, b, \& c$ have default values, they have no effect, so we consider only $d = -1$. Since d causes a vertical shift and the y-axis is negative downward, the only effect we report is "a vertical shift of $d = -1$ units down".

17. We now consider the effects of multiple parameters not in default value. It is nice the total effect can be determined in the order of a, b, c, & d from left to right.

18. Example: $y = \csc(-x+2)$ $\left(= 1 \cdot \csc(-1 \cdot x + 2) + 0\right)$

Initially it seems that $a = 1, b = -1, c = 2, d = 0$. However, all the information provided in this chapter is based on the condition that $b > 0$. This is our first example of dealing with $b < 0$. To correct this situation we apply the negative angle identity $\csc(-\theta) = -\csc\theta$. It is imperative that to use this identity we must take out a factor of -1 from the entire argument of $\csc(-x+2)$, not just from "$-x$". Thus $y = \csc(-x+2) = \csc[-(x-2)] = -\csc(x-2)$, so $y = -\csc(x-2)$ $\left(= -1 \cdot \csc(1x - 2) + 0\right)$.
Then $a = -1, b = 1, c = -2, d = 0$.

Since $|a| = |-1| = 1$, the vertical distance of each point from the x-axis is not changed (when multiplied by $|a| = 1$. However, since $a = -1 < 0$, the graph is reflected through the x-axis.
Since b & d have default values, there is no period change or vertical shift. Finally, $-\frac{c}{b} = -\frac{-2}{1} = +2$, so there is a horizontal shift 2 units (to the right).

19. Example: $y = 3\cos(-2x - \frac{\pi}{6}) - 4$.

Initially we have $a = 3$, $b = -2$, $c = -\frac{\pi}{6}$, $d = -4$. Since $b = -2 < 0$, we use the negative angle identity. Recall that these identities are different for $\cos\theta$ and $\sec\theta$ than the other trig functions. In the last example, we used $\csc(-\theta) = -\csc\theta$. However, $\cos(-\theta) = \cos\theta$, not $-\cos\theta$.

Thus $y = 3\cos(-2x - \frac{\pi}{6}) - 4 = 3\cos[-(2x + \frac{\pi}{6})] - 4 = 3\cos(2x + \frac{\pi}{6}) - 4$, so $a = 3$, $b = 2$, $c = \frac{\pi}{6}$, $d = -4$.

Since $a = 3 > 0$, the amplitude is $|a| = |3| = 3$, and there is no reflection through the x-axis (but we do not need to report "no reflection"). Note that for $\cos\theta$ we can use the easier language of "amplitude" rather than "vertical distance".

Since $\cos\theta$ has a period of 2π, the period here is period $= \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

The horizontal shift is $-\frac{c}{b} = -\frac{\frac{\pi}{6}}{2} = -\frac{\pi}{12}$ units left.

The vertical shift is $d = -4$ units down.

20. <u>Example</u>: $y = \dfrac{\tan\left(\frac{x}{6}+\frac{\pi}{2}\right)}{2} + 3 \left(= \frac{1}{2}\cdot\tan\left(\frac{1}{6}\cdot x + \frac{\pi}{2}\right) + 3\right)$.

Thus $a = \frac{1}{2}, b = \frac{1}{6}, c = \frac{\pi}{2}, d = 3$.

The vertical distance of each point from the x-axis is multiplied by $|a| = \left|\frac{1}{2}\right| = \frac{1}{2}$.

Since $\tan\theta$ has a period of π, the period in this case is period $= \frac{\pi}{b} = \frac{\pi}{\frac{1}{6}} = 6\pi$.

The horizontal shift is $-\frac{c}{b} = -\frac{\frac{\pi}{2}}{\frac{1}{6}} = 3\pi$ units <u>right</u>.

The vertical shift is $d = 3$ units <u>upward</u> (since $d>0$).

21. Example: $y = 5\sin(-3x+6) - 4$.

Since $b = -3 < 0$, we apply the identity $\sin(-\theta) = -\sin\theta$,
so $y = 5\sin(-3x+6) - 4 = 5\sin[-(3x-6)] - 4 =$
$-5\sin(3x-6) - 4$.
Thus $y = -5\sin(3x-6) - 4$, and so
$a = -5$, $b = 3$, $c = -6$, $d = -4$.

Since $a = -5 < 0$, the graph is reflected through the x-axis. Also, since $\sin\theta$ has an amplitude of 1, then the amplitude in this case is amplitude $= |a| = |-5| = 5$.

Since $\sin\theta$ has a period of 2π, then the period in this case is period $= \frac{2\pi}{b} = \frac{2\pi}{3}$.

The horizontal shift is $-\frac{c}{b} = -\frac{-6}{3} = +2$ units right. (Recall that horizontal shifts are in the direction of the x-axis, which is positive to the right. Since $-\frac{c}{b} = 2 > 0$, the shift is to the right.)

The vertical shift is $d = -4$ units down. (Recall that vertical shifts are in the direction of the y-axis, which is negative downward. Since $d = -4 < 0$, the shift is downward.)