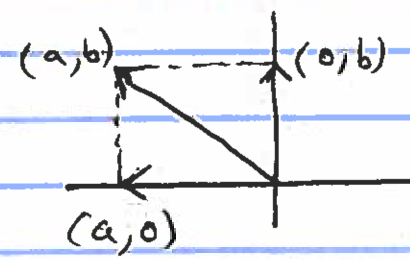
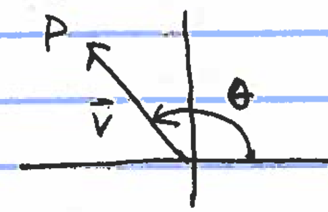
Section 7.5

1. A vector in $\mathbb{R}^2$ from $(0,0)$ to (a,b) can be represented by (a,b). More generally, the vector $\vec{x}$ from (a,b) to (c,d) has the same magnitude and direction as the vector $\vec{y}$ from $(0,0)$ to $(c-a, d-b)$. Therefore $\vec{x} = \vec{y} = (c-a, d-b)$.

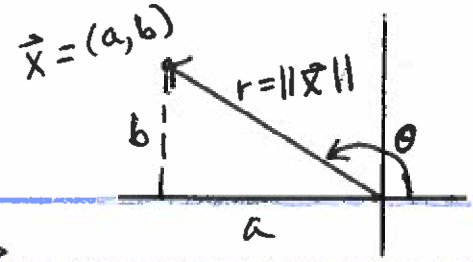
2. Just like the "point" (a,b), the vector $\vec{x} = (a,b)$ has the horizontal component $(a,0)$ and the vertical component $(0,b)$. Thus $(a,0)+(0,b) = (a,b)$.



3. The "direction angle" of a vector $\vec{v}$ from the origin to a point P is the smallest nonnegative angle θ from the positive x-axis to $\vec{v}$.



Note: The textbook says:
   (a) "x-axis" instead of "positive x-axis", and
   (b) "positive angle" instead of "smallest nonnegative angle".
This causes many problems and ambiguities.



4. If $\vec{x}=(a,b)$ has direction angle θ, then $\dfrac{a}{\|\vec{x}\|}=\cos\theta$ and $\dfrac{b}{\|\vec{x}\|}=\sin\theta \Rightarrow$

$a=\|\vec{x}\|\cdot\cos\theta$ and $b=\|\vec{x}\|\cdot\sin\theta$.

Thus $\vec{x}=(a,b)$ has

horizontal component $(a,0)=(\|\vec{x}\|\cdot\cos\theta, 0)$ and

vertical component $(0,b)=(0,\|\vec{x}\|\cdot\sin\theta)$

5. $\underline{\vec{i}\text{-}\vec{j}\text{ Notation}}$: Define $\vec{i}=(1,0)$ & $\vec{j}=(0,1)$.

If $\vec{x}$ is the vector from the origin to the point (a,b), then $\vec{x}=(a,b)=(a,0)+(0,b)=a\cdot(1,0)+b\cdot(0,1)=a\vec{i}+b\vec{j}$.

Example: $\vec{x}=(-7,3)=-7\vec{i}+3\vec{j}$ and

$\vec{y}=4\vec{i}-9\vec{j}=(4,-9)$

6. Application: It is frequently necessary to decompose vectors into the sum of their horizontal and vertical components in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

(a) $\vec{x}=(-3,4)=(-3,0)+(0,4)=-3\cdot(1,0)+4\cdot(0,1)=-3\vec{i}+4\vec{j}$.

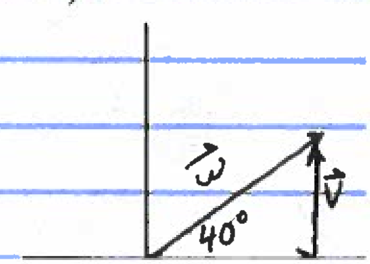
(b) If $\vec{w}$ has direction angle 40° and $\|\vec{w}\|=25$, then

$\|\vec{u}\|=25\cos 40^\circ=19.2$ & $\|\vec{v}\|=25\sin 40^\circ=16.1$.

Thus $\vec{u}=(19.2,0)$ & $\vec{v}=(0,16.1) \Rightarrow$

$\vec{w}=\vec{u}+\vec{v}=(19.2,0)+(0,16.1)=$

$19.2(1,0)+16.1(0,1)=19.2\vec{i}+16.1\vec{j}$.



7. Vector Equality (Various Vector Notations):

If $\vec{x} = (a,b)$ and $\vec{y} = (c,d)$, then
$\vec{x} = \vec{y}$ iff $(a,b) = (c,d)$ iff $a = c$ and $b = d$.

Given points $P(a,b)$, $Q(c,d)$, $S(u,v)$, $T(x,y)$ in $\mathbb{R}^2$,
the vector from P to Q is $\overrightarrow{PQ} = (c-a, d-b)$,
the vector from S to T is $\overrightarrow{ST} = (x-u, y-v)$, and
$\overrightarrow{PQ} = \overrightarrow{ST}$ iff $(c-a, d-b) = (x-u, y-v)$ iff
$c-a = x-u$ and $d-b = y-v$.

In ij-notation, $a\vec{i} + b\vec{j} = c\vec{i} + d\vec{j}$ iff $a = c$ and $b = d$.

8. Vector Sums and Differences (Various Notations):

$(a,b) + (c,d) = (a+c, b+d)$
$(a,b) - (c,d) = (a-c, b-d)$

$(a\vec{i} + b\vec{j}) + (c\vec{i} + d\vec{j}) = (a+c)\vec{i} + (b+d)\vec{j}$
$(a\vec{i} + b\vec{j}) - (c\vec{i} + d\vec{j}) = (a-c)\vec{i} + (b-d)\vec{j}$

9. Negative of a Vector (Various Notations):

$-(a,b) = (-a,-b)$; $-(a\vec{i} + b\vec{j}) = -a\vec{i} - b\vec{j}$

10. Scalar Multiple of a Vector (Various Notations):

$r \cdot (a,b) = (ra, rb)$; $r \cdot (a\vec{i} + b\vec{j}) = (ra)\vec{i} + (rb)\vec{j}$

11. Zero Vector (Various Notations):

$\vec{0} = (0,0) = 0\vec{i} + 0\vec{j}$

12. The "magnitude" of a vector $\vec{x}$ is a measure of the length of $\vec{x}$, denoted $\|\vec{x}\|$.

Various Notations:

$\vec{x} = (a,b) \Rightarrow \|\vec{x}\| = \sqrt{a^2+b^2}$ (distance from $(0,0)$ to (a,b))

If $P(a,b)$ and $Q(c,d)$ are points in $\mathbb{R}^2$, then

$\vec{x} = \overrightarrow{PQ} = (c-a, d-b) \Rightarrow$

$\|\vec{x}\| = \|\overrightarrow{PQ}\| = \sqrt{(c-a)^2+(d-b)^2}$ (distance from (a,b) to (c,d))

If $\vec{x} = a\vec{i} + b\vec{j}$ then $\|\vec{x}\| = \|a\vec{i}+b\vec{j}\| = \sqrt{a^2+b^2}$.

Examples:

$\|(-2,7)\| = \sqrt{(-2)^2+(7)^2} = \sqrt{53}$.

If $P = (-3,1)$ and $Q = (2,5)$ are points in $\mathbb{R}^2$, then

$\|\overrightarrow{PQ}\| = \sqrt{(2-(-3))^2+(5-1)^2} = \sqrt{(5)^2+(4)^2} = \sqrt{41}$.

$\|6\vec{i} - \vec{j}\| = \sqrt{(6)^2+(-1)^2} = \sqrt{37}$.

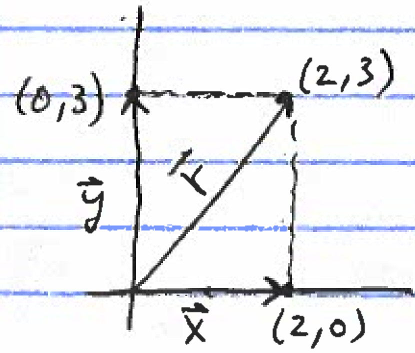
13. If $\vec{x}, \vec{y}, \vec{r}$ are vectors, and $\vec{x}+\vec{y} = \vec{r}$, then $\vec{r}$ is called the "resultant" of $\vec{x}$ and $\vec{y}$, and $\vec{x}$ and $\vec{y}$ are called "components" of $\vec{r}$.

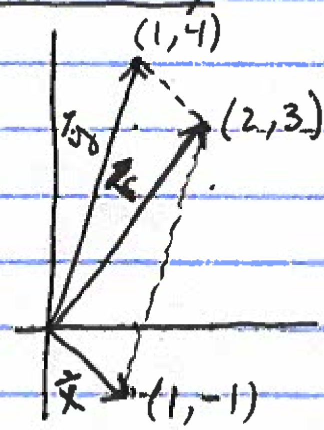
Note: Given $\vec{x}$ & $\vec{y}$, the resultant $\vec{r}$ is unique, Given $\vec{r}$, the components $\vec{x}$ & $\vec{y}$ are NOT unique.

Example: $(2,0)+(0,3) = (2,3)$ & $(1,-1)+(1,4) = (2,3)$.

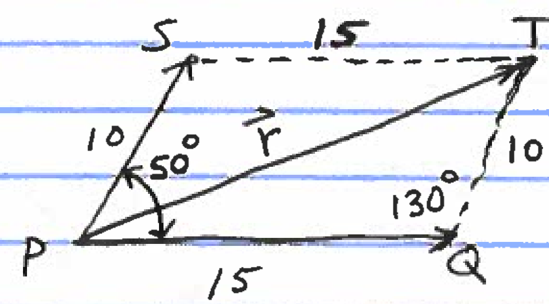
The components of $(2,3)$ are not unique.

the diagrams below explain this algebraic relationship geometrically.





14. Two forces of 10 lb. and 15 lb. act on a point P in $\mathbb{R}^2$. The angle between the forces is 50°. Determine the magnitude of the resultant force $\vec{F}$ on P.



$m\angle P = 50°$ and PQTS is a parallelogram, so $m\angle Q = 180° - 50° = 130°$, $\|\vec{ST}\| = 15$, and $\|QT\| = 10$

By the Law of Cosines in ΔPQT (or ΔPST),

$\|\vec{F}\|^2 = 15^2 + 10^2 - 2(15)(10)\cos 130° \Rightarrow$

$\|\vec{F}\|^2 = 517.8362829 \Rightarrow$

$\|\vec{F}\| = 22.756$ lb.

15. Dot (inner) Product of Vectors $\vec{x}$ & $\vec{y}$:
Notation: $\vec{x} \cdot \vec{y}$

If $\vec{x} = (a, b)$ and $\vec{y} = (c, d)$, then $\vec{x} \cdot \vec{y} = ac + bd$.
If $\vec{x} = a\vec{i} + b\vec{j}$ and $\vec{y} = c\vec{i} + d\vec{j}$, then $\vec{x} \cdot \vec{y} = ac + bd$.
Note: The dot product of two vectors is a number.

16. Examples:
(a) If $\vec{x} = (-2, 3)$, $\vec{y} = (4, 3)$ then $\vec{x} \cdot \vec{y} = (-2)(4) + (3)(3) = 1$.
(b) If $\vec{u} = 12\vec{i} - 5\vec{j}$, $\vec{v} = \frac{1}{2}\vec{i} + 3\vec{j}$ then $\vec{u} \cdot \vec{v} = (12)(\frac{1}{2}) + (-5)(3) = -9$.

17. Application: Suppose a student has exam grades 92, 87, 85 (15% each), midterm grade 79 (25%), and final exam grade 94 (30%). Determine the student's course average ("weighted mean").

Solution: Define the following vectors:
grade vector $\vec{g} = (92, 87, 85, 79, 94)$ and
weight vector $\vec{w} = (15\%, 15\%, 15\%, 25\%, 30\%) =$
$(.15, .15, .15, .25, .30)$.
(converting the percentages to decimal).
Then the student's course average is

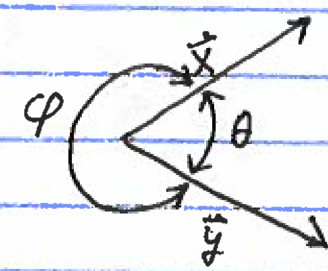
$$\vec{g} \cdot \vec{w} = (92)(.15) + (87)(.15) + (85)(.15) + (79)(.25) + (94)(.30) =$$

$$87.55 \approx 88.$$

18. <u>Theorem</u>: If $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^2$ and θ is the angle between $\vec{x}$ and $\vec{y}$ which satisfies the condition that $0° \le \theta \le 180°$, then

$$\cos\theta = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}, \text{ and so } \theta = \cos^{-1}\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}\right).$$

<u>Note</u>: Since θ satisfies $0° \le \theta \le 180°$, the θ is the <u>smaller</u> angle between $\vec{x}$ and $\vec{y}$, <u>not</u> the larger angle φ.



<u>Note</u>: The reason that θ satisfies $0° \le \theta \le 180°$ is that $\theta = \cos^{-1}\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}\right)$, and the inverse cosine function has the principle value range of $0°$ to $180°$.

19. Find the angle θ between $\vec{u} = 3\vec{i} + \vec{j}$ and $\vec{v} = 4\vec{i} - 2\vec{j}$.

$$\vec{u} \cdot \vec{v} = (3)(4) + (1)(-2) = 10.$$

$$\|\vec{u}\| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}.$$

$$\|\vec{v}\| = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}.$$

$$\therefore \cos\theta = \frac{10}{\sqrt{10} \cdot 2\sqrt{5}} = \frac{10}{2\sqrt{50}} = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ \text{ (from a calculator \underline{or} table).}$$

20. Find the angle θ between $\vec{x} = (4,1)$ and $\vec{y} = (-5,2)$.

$$\vec{x} \cdot \vec{y} = (4)(-5) + (1)(2) = -18.$$

$$\|\vec{x}\| = \sqrt{(4)^2 + (1)^2} = \sqrt{17}.$$

$$\|\vec{y}\| = \sqrt{(-5)^2 + (2)^2} = \sqrt{29}.$$

$$\therefore \cos\theta = \frac{-18}{\sqrt{17} \cdot \sqrt{29}} = \frac{-18}{\sqrt{493}} \Rightarrow$$

$$\theta = \cos^{-1}\left(\frac{-18}{\sqrt{493}}\right) \approx 144.16^\circ \text{ (from a calculator).}$$

<u>Note</u>: It is advisable to multiply $17 \cdot 29 = 493$ and replace $\sqrt{17} \cdot \sqrt{29}$ with $\sqrt{493}$ before computing θ with the inverse cosine on your calculator. This is easy with your calculator and will minimize the chance of an error since the calculation would require $\sqrt{17} \cdot \sqrt{29}$ to be enclosed in parentheses!