Section 7.3

1. Review the information below:

Facts About Triangles

1. For each angle θ in any triangle, 0° < m(∠θ) < 180°.

2. The sum of the angles of any triangle equals 180°.

3. In any triangle ΔABC,
   with angles ∠A, ∠B, and ∠C and
   corresponding sides a, b, and c,
   a < b < c if and only if m(∠A) < m(∠B) < m(∠C).

   In other words, in any triangle
   the shortest side is opposite the smallest angle,
   the medium length side is opposite the medium size angle,
   the longest side is opposite the largest angle.

4. A triangle has **at most** one angle of measure greater than or equal to 90°.

5. If a triangle has an angle of measure greater than or equal to 90°,
   it must be the largest angle (see #4 above), and
   therefore must be opposite the longest side (see #3 above).

2. The Law of Cosines: In ΔABC,

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A,$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B, \text{ and}$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C.$$

In other words, in ΔABC, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of the other two sides multiplied by the cosine of the angle between the other two sides.

<u>Note</u>: If the angle involved is 90°, then $\cos 90° = 0$, so the last term vanishes, reducing the equation to the Pythagorean Theorem for right triangles.

3. The Law of Cosines is needed for solving triangles in the SAS and SSS cases.

Solving Triangles Using the
Law of Sines and Law of Cosines

1. Solving Triangles by the ASA or SAA Methods

Note: One side and two angles are given.

1. Find the third angle by subtracting the two given angles from 180°.
2. Use the given side, the corresponding angle, and either of the two remaining angles to find a second side with the Law of Sines.
3. Use either of the two known sides, the corresponding angle, and the angle opposite the unknown side to find the third side with the Law of Sines.

2. Solving Triangles by the SAS Method

Note: Two sides and the included angle are given

1. Use the Law of Cosines to find the third side.
2. Use the Law of Sines to find the remaining angle opposite the smaller corresponding side.

   Note: Since this side is not the longest side, then the corresponding angle must be acute, thus avoiding the ambiguous case of the Law of Sines.

3. Find the last angle by subtracting the other two from 180°.

3. Solving Triangles by the SSS Method

Note: Three sides are given

1. Use the Law of Cosines to find the angle opposite the longest side.

   Note: If the triangle has an angle of 90° or more, it has only one such angle, and it must be opposite the longest side. Since $\cos^{-1}$ has a range of 0° to 180°, the Law of Cosines will identify such an angle.

2. Use the Law of Sines to find either one of the remaining angles.

   Note: Since the largest angle was found in step 1, the other two angles must be acute, thus avoiding the ambiguous case of the Law of Sines.

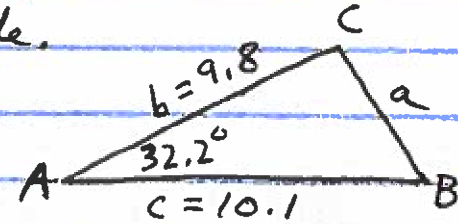
3. Find the last angle by subtracting the other two from 180°.

4. Solving a triangle in the SAS case:

In $\triangle ABC$, suppose $b=9.8$, $c=10.1$, $\angle A=32.2°$

First we determine the third side.

Using the Law of Cosines, we have



$a^2=b^2+c^2-2bc\cdot\cos A \Rightarrow a^2=9.8^2+10.1^2-2(9.8)(10.1)\cos 32.2°$

$\Rightarrow a^2=30.53760083 \Rightarrow a=5.526$ (rounded off).

Next we determine the remaining angle opposite the smaller corresponding side.

Since $b<c$, then we determine $\angle B$ next.

Using the Law of Sines, we have

$$\frac{\sin B}{b}=\frac{\sin A}{a} \Rightarrow \frac{\sin B}{9.8}=\frac{\sin 32.2°}{5.526} \Rightarrow \sin B=\frac{9.8\cdot\sin 32.2°}{5.526} \Rightarrow$$

$\sin B=.9450212641 \Rightarrow B=\sin^{-1}(.9450212641) \Rightarrow$

$B=70.9°$ (rounded).

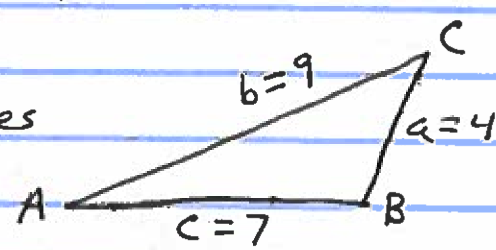
Note: Although $\sin 109.1° = \sin(180°-70.9°)=\sin 70.9°$, we know $\angle B<90°$ since a triangle can have at most 1 angle $\geq 90°$ and it must be opposite the longest side. Since $b<c$ then $\angle B<90°$.

Finally, $\angle C=180°-32.2°-70.9° \Rightarrow \angle C=76.9°$.

5. Note: We could have used the Law of Cosines to solve for $\angle B$ and $\angle C$ in either order. Since $0^\circ \leq \cos^{-1}\theta \leq 180^\circ$, then no ambiguity can occur. However, that approach would be more computationally intensive.

6. Solving a triangle in the SSS case:

In $\triangle ABC$, suppose $a=4, b=9, c=7$:



First we use the Law of Cosines to determine the angle opposite the longest side. If $\triangle ABC$ has an angle $\geq 90^\circ$, it must be opposite the longest side. Unlike $\sin^{-1}\theta$, $0^\circ \leq \cos^{-1}\theta \leq 180^\circ$, so the Law of Cosines can determine an angle $\geq 90^\circ$ with no ambiguity. Since b is the longest side, then we proceed as follows:

$$b^2 = a^2 + c^2 - 2ac \cdot \cos B \Rightarrow 2ac \cdot \cos B = a^2 + c^2 - b^2 \Rightarrow$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{4^2+7^2-9^2}{2 \cdot 4 \cdot 7} = -0.2857142857 \Rightarrow$$

$$B = \cos^{-1}(-0.2857142857) = 106.6^\circ \text{ (rounded)}.$$

Next we determine either one of the remaining angle with the Law of Sines since it is computationally easier than the Law of Cosines. Therefore we choose arbitrarily to determine $\angle A$ next. Thus

$$\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \sin A = \frac{a \cdot \sin B}{b} \Rightarrow$$

$$\sin A = \frac{4 \cdot \sin 106.6^\circ}{9} = 0.4259211442 \Rightarrow$$

$$A = \sin^{-1}(0.4259211442) = 25.2^\circ \text{ (rounded)}.$$

Note: Since $\angle B$ is opposite the longest side, and a triangle cannot have more than one angle $\geq 90^\circ$, then $\angle A < 90^\circ$ and $\angle C < 90^\circ$. Thus even though $\sin 154.8^\circ = \sin(180^\circ - 25.2^\circ) = \sin 25.2^\circ$, we know that $\angle A = 25.2^\circ$ (not 154.8°).

Finally, $\angle C = 180^\circ - 106.6^\circ - 25.2^\circ \Rightarrow \angle C = 48.2^\circ$.

Note: We could have determined $\angle C$ with either the Law of Cosines or the Law of Sines. However, these are clearly much more computationally intensive than the approach taken here.

7. Previously we saw that the area of ΔABC is
$$\text{Area} = \tfrac{1}{2}ab\cdot\sin C = \tfrac{1}{2}ac\cdot\sin B = \tfrac{1}{2}bc\cdot\sin A.$$

The area of ΔABC can also be obtained using only the sides a, b, c.

<u>Heron's Formula</u>:

For any ΔABC,

the semiperimeter is $s = \frac{1}{2}\cdot(a+b+c)$, and

the area is $\text{Area} = \sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-c)}$.

8. Example: Suppose ΔABC has sides
$a = 7.2$ ft, $b = 4.9$ ft, and $c = 7.8$ ft.

Then the semiperimeter is
$$s = \tfrac{1}{2}\cdot(7.2\text{ ft} + 4.9\text{ ft} + 7.8\text{ ft}) = \tfrac{1}{2}\cdot(19.9\text{ ft}) = 9.95\text{ ft}.$$

Thus the area of ΔABC is

$$\text{Area} = \sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-c)} =$$

$$\sqrt{9.95\text{ ft}\cdot(9.95-7.2)\text{ ft}\cdot(9.95-4.9)\text{ ft}\cdot(9.95-7.8)\text{ ft}} =$$

$$\sqrt{297.236250861\text{2 ft}^4} = 17.24\text{ ft}^2.$$

9. A triangular yard with sides of length 680 ft, 350 ft, and 920 ft must be fertilized. A bag of fertilizer covers 650 ft^2. How many bags are required to fertilize the yard?

Solution: The semiperimeter of the yard is

$$s = \frac{1}{2}\cdot(680\text{ ft} + 350\text{ ft} + 920\text{ ft})$$

$$= \frac{1}{2}\cdot(1950\text{ ft}) = 975\text{ ft}.$$

The area of the yard is

$$\text{Area} = \sqrt{s\cdot(s-a)\cdot(s-b)\cdot(s-c)} =$$

$$\sqrt{975\text{ ft}\cdot(975-680)\text{ ft}\cdot(975-350)\text{ ft}\cdot(975-920)\text{ ft}} =$$

$$\sqrt{\cancel{99433.9447825\text{ ft}}} = 99433.9447825\text{ ft}^2.$$

Since each bag covers 650 ft^2, then the number of bags required to fertilize the yard is:

$$\frac{\text{Area}}{650} = \frac{99433.9447825\text{ ft}^2}{650\text{ ft}^2/\text{bag}} = 152.975\text{ bags}.$$

Since partial bags are not sold, then buy 153 bags.