Sections 5.3 - 5.4

1. This material is devoted to the Sum & Difference Identities, which are listed below.

Sum and Difference Identities

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$$

$$\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$\cot(A+B) = \frac{1 - \tan A \cdot \tan B}{\tan A + \tan B}$$

$$\cot(A-B) = \frac{1 + \tan A \cdot \tan B}{\tan A - \tan B}$$

$$\sec(A+B) = \frac{\sec A \cdot \sec B}{1 - \tan A \cdot \tan B}$$

$$\sec(A-B) = \frac{\sec A \cdot \sec B}{1 + \tan A \cdot \tan B}$$

$$\csc(A+B) = \frac{\csc A \cdot \csc B}{\cot A + \cot B}$$

$$\csc(A-B) = -\frac{\csc A \cdot \csc B}{\cot A - \cot B} = \frac{\csc B \cdot \csc A}{\cot B - \cot A}$$

Note: Some of these may not appear in your textbook.

2. When verifying identities with the Sum and Difference Identities, apply the same rules as outlined by the hints in the preceding section.

3. Example: Verify $\dfrac{\sin(A+B)}{\sin A \cdot \sin B} = \cot A + \cot B$.

Starting with the more complex left side,

$$\frac{\sin(A+B)}{\sin A \sin B} =$$

$$\frac{\sin A \cos B + \cos A \sin B}{\sin A \sin B} \text{ (hint 1; Sum/Diff identities)} =$$

$$\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B} \text{ (hint 5; right side has 2 terms) (hint 4; rules of algebra)} =$$

$$\frac{\cos B}{\sin B} + \frac{\cos A}{\sin A} \text{ (hint 4; rules of algebra)} =$$

$$\cot B + \cot A \text{ (hint 1; quotient identities)} =$$

$$\cot A + \cot B \text{ (hint 4; rules of algebra)}.$$

Hence $\dfrac{\sin(A+B)}{\sin A \sin} = \cot A + \cot B$.

4. Identities are useful not only for verifying other identities, but also for computational purposes when <u>exact</u> answers are required. Combined with the Table of Exact Values, the following examples illustrate this.

5.

Table of Exact Trigonometric Values

θ (rad / deg)	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞
$\csc\theta$	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1

6. Compute an exact value of $\cos 15°$.

$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° =$

$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$ (order of operations) $= \frac{\sqrt{6} + \sqrt{2}}{4}$.

Notes: The cosine function in $\cos 15°$ immediately narrowed down the list of 13 Sum/Diff identities to the 3rd and 4th identities for $\cos(A+B)$ and $\cos(A-B)$. Since $15°$ was written as the difference $45° - 30°$ (rather than as a sum of two angles), then we knew to use $\cos(A-B) = \cos A \cos B + \sin A \sin B$. Also, the order of operations required the two multiplications to be done before the addition in $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$.

7. Compute an <u>exact</u> value of $\tan 75°$.

$$\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \cdot \tan 30°} =$$

$$\frac{1 + \frac{\sqrt{3}}{3}}{1 - 1 \cdot \frac{\sqrt{3}}{3}} \text{ (Table of Exact Values)} =$$

$$\frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} \text{ (order of operations; mult. befor add.)} =$$

$$\frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} \cdot \frac{3}{3} = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{3 + \sqrt{3}}{3 + \sqrt{3}} \left(\begin{array}{l}\text{to rationalize the} \\ \text{denominator}\end{array}\right) =$$

$$\frac{9 + 3\sqrt{3} + 3\sqrt{3} + 3}{9 + 3\sqrt{3} - 3\sqrt{3} - 3} = \frac{12 + 6\sqrt{3}}{6} = \frac{6(2 + \sqrt{3})}{6} = 2 + \sqrt{3}.$$

<u>Notes</u>: In $1 - 1 \cdot \frac{\sqrt{3}}{3}$, the multiplication must be done <u>before</u> the subtraction as required by the normal order of arithmetic operations.

Although it might have been tempting to stop with $\frac{3 + \sqrt{3}}{3 - \sqrt{3}}$, we see that rationalizing the denominator resulted in a much simpler expression $2 + \sqrt{3}$. While there is no crime in leaving a radical in a denominator (I was always told this was "forbidden"), the simpler answer makes the extra work worthwhile.

8. <u>NOTE</u>: Be prepared to do these types of computations in <u>RADIANS</u>!

9. Compute <u>an exact</u> value of $\sec\frac{\pi}{12}$.
<u>Note</u>: No degree symbol implies <u>radians</u>.

$$\sec\frac{\pi}{12} = \sec\frac{3\pi-2\pi}{12} = \sec\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right) = \sec\left(\frac{\pi}{4}-\frac{\pi}{6}\right) =$$

$$\frac{\sec\frac{\pi}{4}\cdot\sec\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}} =$$

$$\frac{\sqrt{2}\cdot\frac{2\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} \quad \left(\begin{array}{l}\text{Table of Exact Values}\\ \text{Note: } \frac{\pi}{12} \text{ is not in the Table}\end{array}\right) =$$

$$\frac{\frac{2\sqrt{6}}{3}}{1+\frac{\sqrt{3}}{3}} \quad (\text{order of operations; mult. before add.}) =$$

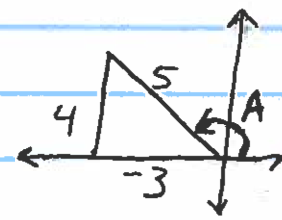
$$\frac{\frac{2\sqrt{6}}{3}}{1+\frac{\sqrt{3}}{3}}\cdot\frac{3}{3} = \frac{2\sqrt{6}}{3+\sqrt{3}}\cdot\frac{3-\sqrt{3}}{3-\sqrt{3}} \quad \left(\begin{array}{l}\text{to rationalize the}\\ \text{denominator}\end{array}\right) =$$

$$\frac{6\sqrt{6}-2\sqrt{18}}{9-3\sqrt{3}+3\sqrt{3}-3} = \frac{6\sqrt{6}-2\cdot 3\sqrt{2}}{6} = \frac{6\sqrt{6}-6\sqrt{2}}{6} =$$

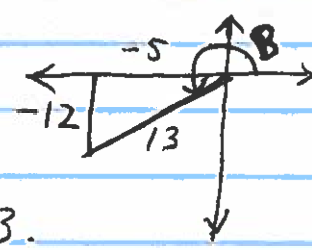
$$\frac{6\cdot(\sqrt{6}-\sqrt{2})}{6} = \sqrt{6}-\sqrt{2}.$$

10. We now return to a familiar problem, modified to use the Sum/Diff Identities

11. Compute an exact value of $\tan(A+B)$ if $\sin A = \frac{4}{5}$, $\cos B = -\frac{5}{13}$, $\cos A < 0$, $\sin B < 0$.



(a) $\sin A = \frac{4}{5} > 0$ & $\cos A < 0 \Rightarrow A$ is in QII.
$\sin A = \frac{4}{5} = \frac{opp}{hyp} \Rightarrow$ y-side $= 4$, hyp $= 5$
Using the Pythagorean Theorem for the x-side, $x^2 + (4)^2 = (5)^2 \Rightarrow x^2 + 16 = 25 \Rightarrow x^2 = 9 \Rightarrow x = \pm\sqrt{9} = \pm 3$. The diagram indicates that the x-side is in the negative direction, so $x = -3$.



(b) $\cos B = -\frac{5}{13} < 0$ & $\sin B < 0 \Rightarrow B$ is in QIII.
$\cos B = -\frac{5}{13} = \frac{adj}{hyp} \Rightarrow$
x-side has length 5 & hyp has length 13.
Since the hyp-side is always positive, then the x-side gets the minus sign. Therefore $x = -5$ and hyp $= 13$. Then $x^2 + y^2 = r^2 \Rightarrow$
$(-5)^2 + y^2 = (13)^2 \Rightarrow 25 + y^2 = 169 \Rightarrow y^2 = 144 \Rightarrow y = \pm\sqrt{144} = \pm 12$.
The diagram shows the y-side in the negative direction, and so $y = -12$.

(c) Then $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} = \frac{-\frac{4}{3} + \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)} = \frac{\frac{16}{15}}{\frac{63}{15}} =$

$\frac{16}{15} \cdot \frac{15}{63} = \frac{16}{63}$. (Note that $\tan A$ & $\tan B$ were obtained using the diagrams for $\angle A$ & $\angle B$ separately.)